Complex-Distance Potential Theory and Hyperbolic Equations

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February 4, 2008

Abstract

An extension of potential theory in $\mathbb{R}^n$ is obtained by continuing the Euclidean distance function holomorphically to $\mathbb{C}^n$. The resulting Newtonian potential is generated by an extended source distribution $\tilde{\delta}(z)$ in $\mathbb{C}^n$ whose restriction to $\mathbb{R}^n$ is the point source $\delta(x)$. This provides a possible model for extended particles in physics. In $\mathbb{C}^{n+1}$, interpreted as complex *spacetime*, $\tilde{\delta}$ acts as a *propagator* generating solutions of the wave equation from their initial values. This gives a new connection between elliptic and hyperbolic equations that does not assume analyticity of the Cauchy data. Generalized to Clifford analysis, it induces a similar connection between solutions of elliptic and hyperbolic Dirac equations. There is a natural application to the time-dependent, inhomogeneous Dirac and Maxwell equations, and the 'electromagnetic wavelets' introduced previously are an example.

1   Motivation and Preliminaries

Most fundamental theories of physics are based on the concept of potentials and fields generated by *point sources,* which presupposes that objects or "particles" can, in principle, be localized within arbitrarily small regions of space and/or time. This is a vast extrapolation from empirical evidence, and it should perhaps not come as a surprise if such theories experience some fundmental difficulties. In Newtonian mechanics, the problem of N point "bodies" interacting through gravitation has, in general, no solution due to the possibility of collisions. This becomes a serious difficulty for $N \geq 3$, where the set of initial conditions leading to collisions is nontrivial [AM78, Chapter 10]. In classical electrodynamics, difficulties arise where the field produced by a point charge unavoidably acts back on the same charge, leading to infinite self-energies and run-away particle trajectories [J99]. A way out of this dilemma was proposed by Wheeler and Feynman [WF45, WF49], but their *action-at-a distance* theory, apart from being highly counter-intuitive [F64, p. 28-8], has resisted quantization and is not generally regarded as being a fundamental description of Nature. In quantum electrodynamics and other quantum field theories, point particles cause divergences which necessitate infinite "renormalization" procedures, a subject of some contraversy [C99]. String theory [P98] does, in fact, not need infinite renormalization because its basic objects (strings) are extended in space rather than mathematical points. This is one of the reasons it is regarded with great hope as a possibility for unifying physical theories. However, a full development of (super)string physics is rather difficult and not expected to near completion for many years.

The ideas developed here began many years ago, motivated in part by the hope that an extension of physics to *complex spacetime,* justified at the foundational level, might give a way to circumvent the problems associated with point sources by applying residue methods. After some years of study and research this led to papers [K77, K78, K80, K87] and books [K90, K94] whose main thrust has been to develop a direct physical interpretation of the complex spacetime as an *extended phase space,* with the imaginary spacetime parameters carrying directional information while the real spacetime parameters describe (approximate!) localization. A major stumbling block in this program has been the construction of extended sources, since it seemed that they tend to spoil the holomorphy of the theory globally rather than just locally. Here we propose a natural solution to this problem. It turns out that although holomorphy enters the theory at the

level of the fundamental potential or Green's function and this indeed gives a canonical extension of general fields to complex spacetime through convolution, these fields need not be holomorphic *anywhere* due to the specific structure of the extended sources.

We begin in Section 2 by extending the Euclidean distance $r(\boldsymbol{x})$ to a function $\gamma(\boldsymbol{x} + i\boldsymbol{y})$ holomorphic in a domain $\mathcal{O}_n \subset \mathbb{C}^n$. This leads to a natural coordinate system in $\mathbb{R}^n$ which will play an important role in the sequel: the *oblate spheroidal coordinates* adapted to $\boldsymbol{y} \neq \boldsymbol{0}$.

In Sections 3 and 4 we use $\gamma(\boldsymbol{z})$ to extend the Newtonian potential $\phi(\boldsymbol{x})$ (fundamental solution of the Laplacian in $\mathbb{R}^n$) to $\mathcal{O}_n$. The resulting *holomorphic potential* $\phi(\boldsymbol{z})$ has a source distribution $\tilde{\delta}(\boldsymbol{z})$ in $\mathbb{C}^n$ which, although nowhere holomorphic, is nevertheless a canonical extension to $\mathbb{C}^n$ of the point source $\delta(\boldsymbol{x})$ in $\mathbb{R}^n$. For *even* $n \geq 4$, $\tilde{\delta}(\boldsymbol{x} + i\boldsymbol{y})$ with fixed $\boldsymbol{y} \neq \boldsymbol{0}$ is supported in $\boldsymbol{x}$ on the *sphere* $\mathcal{B}(\boldsymbol{y})$ of codimension 2 and radius $|\boldsymbol{y}|$, centered at the origin and lying in the hyperplane orthogonal to $\boldsymbol{y}$. For all other values of n, it is supported in $\boldsymbol{x}$ on a *membrane* whose boundary is $\mathcal{B}(\boldsymbol{y})$. The membrane is determined by a choice of branch cut in γ, and in the simplest case it is a *disk* $E_0(\boldsymbol{y})$.

In Section 5 we compute the sources $\tilde{\delta}(\boldsymbol{x} + i\boldsymbol{y})$ explicitly as distributions in $\mathbb{R}^3$ and $\mathbb{R}^4$ supported, respectively, in $E_0(\boldsymbol{y})$ and $\mathcal{B}(\boldsymbol{y})$. This is especially interesting in the physical case of $\mathbb{R}^3$, where $\tilde{\delta}(\boldsymbol{x} + i\boldsymbol{y})$ consists of a uniform line charge on the rim of the disk $E_0(\boldsymbol{y})$ accompanied by a simple layer (with zero net charge) and a double layer distributed on the interior. Hence the original unit "charge" carried by $\delta(\boldsymbol{x})$ is now spread over a circle of radius $|\boldsymbol{y}|$, and a variable polarization density is induced on the disk swept out by "blowing up" the point source from the origin to its rim. That so much natural structure results from a simple analytic continuation is remarkable.

In Section 6 we show how to include *time* in this framework. Since we already have a holomorphic extension $\phi(\boldsymbol{x}, s + it)$ of the fundamental solution $\phi(\boldsymbol{x}, s)$ of the Laplacian in $\mathbb{R}^{n+1}$, it is natural to interpret t as a time parameter and look for a connection with the wave operator $\Delta_{\boldsymbol{x}} - \partial_t^2$, which is the "analytic continuation" to $\mathbb{R}^{n,1}$ of the Laplacian $\Delta_{\boldsymbol{x}} + \partial_s^2$ in $\mathbb{R}^{n+1}$, now interpreted as Euclidean spacetime. We show that $\tilde{\delta}(\boldsymbol{x}, s + it)$ acts as a *propagator,* generating solutions of the initial-value problem for the wave equation in $\mathbb{R}^{n,1}$ from Cauchy data at $t = 0$. This gives a connection between the Laplace equation and the wave equation without assuming analyticity of the initial data, because the propagator $\tilde{\delta}(\boldsymbol{x}, s + it)$ is not holomorphic in $s + it$.

In Section 7 we extend our method to Clifford analysis, where the Laplacian and the wave operator are replaced by their "square roots," the *elliptic and hyperbolic Dirac operators.* The time-dependent Maxwell equations are a natural application of the ensuing formalism.

The idea of extending harmonic functions to obtain solutions of the wave equation has been studied by Garabedian [G64] and, in a more general context, by Ryan [R90, R90a, R96a]. However, the methods employed in these references depend critically on the assumption that the boundary data for the harmonic functions is holomorphic. As emphasized above, our method is free of this restriction.

2   Complex Distance and Spheroidal Coordinates

For $n \geq 2$, let

$$\boldsymbol{z} = \boldsymbol{x} + i\boldsymbol{y} \in \mathbb{C}^n, \qquad |\boldsymbol{x}| = r, \ |\boldsymbol{y}| = a. \tag{1}$$

Define the *complex length* of $\boldsymbol{z}$ as

$$\gamma(\boldsymbol{z}) \equiv \sqrt{\boldsymbol{z}^2} = \sqrt{\boldsymbol{x}^2 - \boldsymbol{y}^2 + 2i\boldsymbol{x} \cdot \boldsymbol{y}} = \sqrt{r^2 - a^2 + 2i\boldsymbol{x} \cdot \boldsymbol{y}} \equiv p + iq. \tag{2}$$

The *complex distance* between any two points $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathbb{C}^n$ is then defined as $\gamma(\boldsymbol{z}_1 - \boldsymbol{z}_2)$. To complete this definition, a branch of the complex root must be chosen. The branch points form the *null cone*

$$\mathcal{N} = \{\boldsymbol{z} \mid \gamma(\boldsymbol{z}) = 0\} = \left\{\boldsymbol{x} + i\boldsymbol{y} \mid \boldsymbol{x}^2 = \boldsymbol{y}^2 \text{ and } \boldsymbol{x} \cdot \boldsymbol{y} = 0\right\}, \tag{3}$$

a manifold of real dimension $2n - 2$ in $\mathbb{C}^n \approx \mathbb{R}^{2n}$. In the context of the wave equation, $\mathcal{N}$ will be seen to be related to the *light cone.* We will be interested in the restriction of $\mathcal{N}$ to a fixed nonzero $\boldsymbol{y} \in \mathbb{R}^n$, *i.e.,*

$$\mathcal{B} \equiv \mathcal{B}(\boldsymbol{y}) = \{\boldsymbol{x} \in \mathbb{R}^n \mid r = a, \ \boldsymbol{x} \cdot \boldsymbol{y} = 0\}. \tag{4}$$

$\mathcal{B}$ is the *sphere* of dimension $n - 2$ in $\mathbb{R}^n$ obtained by intersecting the sphere $r = a$ with the hyperplane orthogonal to $\boldsymbol{y}$. We call $\mathcal{B}(\boldsymbol{y})$ the *branching sphere with axis vector $\boldsymbol{y}$*.

We now concentrate on $\gamma(\boldsymbol{x} + i\boldsymbol{y})$ as a function of $\boldsymbol{x}$, regarding $\boldsymbol{y}$ as a fixed nonzero vector. In order to make this function single-valued, it is necessary to introduce a branch cut in $\mathbb{R}^n$. To see how this can be done, fix any $\boldsymbol{x} \in \mathcal{B}$. In the plane determined by the two orthogonal vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, draw a circle of radius $\varepsilon \leq a$ centered at $\boldsymbol{x}$. Each time we go around this circle, γ changes sign. In order to obtain a single-valued function for γ, it is therefore necessary to cut every such circle. Furthermore, this must be done subject to the requirement that γ must be an extension to $\mathbb{C}^n$ of the usual length function in $\mathbb{R}^n$, i.e.,

$$\boldsymbol{y} \to \boldsymbol{0} \;\Rightarrow\; \gamma(\boldsymbol{x} + i\boldsymbol{y}) \to r(\boldsymbol{x}) \geq 0. \tag{5}$$

The simplest cut (but certinainly not the only one) is obtained by requiring that

$$p \equiv \operatorname{Re} \gamma \geq 0, \tag{6}$$

which means that each of the above circles is cut at the point

$$\boldsymbol{x}_C = (a - \varepsilon)\hat{\boldsymbol{x}}, \qquad \hat{\boldsymbol{x}} \equiv \boldsymbol{x}/r.$$

The totality of such points, as $\boldsymbol{x}$ varies over $\mathcal{B}$ and $0 < \varepsilon \leq a$, together with $\mathcal{B}$ itself ($\varepsilon = 0$), forms the set

$$E_0 \equiv E_0(\boldsymbol{y}) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \mid r \leq a, \ \boldsymbol{x} \cdot \boldsymbol{y} = 0 \right\}, \tag{7}$$

which is the *disk* of dimension $n - 1$ in $\mathbb{R}^n$ obtained by intersecting the ball $r \leq a$ with the hyperplane through the origin orthogonal to $\boldsymbol{y}$. The boundary or *rim* of E_0 is $\mathcal{B}$:

$$\partial E_0 = \mathcal{B}.$$

(For $n = 3$, E_0 is indeed the disk of radius a orthogonal to $\boldsymbol{y}$ and $\mathcal{B}$ is its rim.) In the context of holomorphic potential theory, E_0 or $\mathcal{B}$, depending on whether n is odd or even, will represent the support of a source distribution. As $\boldsymbol{y} \to \boldsymbol{0}$, both sets contract to the origin, the support of the usual point source.

We now derive the basic properties of the real and imaginary parts p, q of γ, which will be important in our study of the holomorphic potentials. From (2) it follows that

$$p^2 - q^2 = r^2 - a^2, \qquad pq = \boldsymbol{x} \cdot \boldsymbol{y} \equiv a\zeta, \tag{8}$$

where ζ is the projection of $\boldsymbol{x}$ onto $\boldsymbol{y} \neq \boldsymbol{0}$. Define the cylindrical coordinate ρ by

$$\rho^2 = r^2 - \zeta^2 = a^2 + p^2 - q^2 - \frac{p^2 q^2}{a^2} = \frac{(a^2 + p^2)(a^2 - q^2)}{a^2}. \tag{9}$$

This shows that the imaginary part of γ is bounded by $a = |\boldsymbol{y}|$:

$$-a \leq q \leq a. \tag{10}$$

We will use (ρ, ζ) as part of a cylindrical coordinate system in $\mathbb{R}^n$ with $\boldsymbol{y}$ as its "z-axis." Equations (8) and (9) show that the surface $E_p = \{\boldsymbol{x} \mid p = \text{constant}\}$ with a given value of $p > 0$ is an *oblate spheroid* given by

$$E_p: \qquad \frac{\rho^2}{a^2 + p^2} + \frac{\zeta^2}{p^2} = 1. \tag{11}$$

Similarly, the level surface $\{\boldsymbol{x} \mid q^2 = \text{constant}\}$ with a given value of $0 < q^2 < a^2$ is a *hyperboloid of one sheet* given by

$$H_q: \qquad \frac{\rho^2}{a^2 - q^2} - \frac{\zeta^2}{q^2} = 1. \tag{12}$$

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Note that we have so far avoided the surfaces with $p = 0$ and $q = 0, \pm a$. These give degenerate forms of E_p and H_q. In fact, as $p \to +0$, the oblate spheroid E_p contracts to the disk (7):

$$p \to 0 \ \Rightarrow \ E_p \to \{\boldsymbol{x} \mid \zeta = 0, \ \rho \le a\} = E_0. \tag{13}$$

More precisely, since the interior of E_0 is covered twice, we will distinguish between its *front and back sides:*

$$E_0 = E_0^+ \cup E_0^- \cup \mathcal{B}, \tag{14}$$

where

$$E_0^{\pm} = \{\boldsymbol{x} \mid \zeta = \pm 0, \ \rho < a\} = \{\boldsymbol{x} \mid p = 0, \ \pm q > 0\} \tag{15}$$

are the *interiors* of the front and back sides of E_0. Although E_0^+ and E_0^- coincide as sets, the distinction between them will be very important for the following reason. It can be easily seen that

$$\boldsymbol{x} \in E_0^{\pm} \ \Rightarrow \ q = \pm\sqrt{a^2 - r^2} = \pm\sqrt{a^2 - \rho^2}. \tag{16}$$

This shows that while p is continuous across E_0, q has a spherical jump discontinuity there. In the context of holomorphic potential theory, E_0^+ and E_0^- will be seen to form the two sides of a *double layer.*

The degenerate values of q are $q = 0, \pm a$. As $q \to \pm a$, the semi-hyperboloids $H_q^{\pm}$ with $\pm q > 0$ collapse to the positive and negative ζ-axis, respectively. As $q \to 0$, H_q collapses to the set $H_0 = \{\boldsymbol{x} \mid \zeta = 0, \ \rho \ge a\}$ which, like E_0, is covered twice, once by the limit of each semi-hyperboloid $H_q^{\pm}$ as $q \to \pm 0$. But in this case we do not distinguish between the two copies because γ is continuous across H_0.

As mentioned above, the parameters ζ and ρ are part of a cylindrical coordinate system in $\mathbb{R}^n$. Note that the intersection of E_p and H_q is

$$S_{\gamma} = E_p \cap H_q = \left\{\boldsymbol{x} \in \mathbb{R}^n \mid a\zeta = pq, \ a\rho = \sqrt{a^2 + p^2}\sqrt{a^2 - q^2}\right\}.$$

Here and henceforth, we denote the pair of real variables (p, q) by the single complex variable $\gamma = p + iq$ for convenience. If $q \ne \pm a$, then S_{γ} is a sphere of dimension $n - 2$ and radius ρ. The variable point on S_{γ} may be represented as

$$\boldsymbol{x} = \rho\boldsymbol{\sigma} + \zeta\hat{\boldsymbol{y}} \in S_{\gamma}, \tag{17}$$

where $\boldsymbol{\sigma}$ runs over the unit sphere of dimension $n - 2$ in the hyperplane orthogonal to $\boldsymbol{y}$. In particular, $\gamma = 0$ gives $\zeta = 0$ and $\rho = a$, so that

$$S_0 = \{a\boldsymbol{\sigma} \mid \boldsymbol{\sigma} \in S^{n-2}\} = \mathcal{B}(\boldsymbol{y}).$$

If $n = 2$, S_{γ} consists of just two points. If $n = 3$, S_{γ} is a circle. Thus, for fixed $\boldsymbol{y} \ne \boldsymbol{0}$, a complete set of *cylindrical coordinates* in $\mathbb{R}^n$ is given by $(\rho, \zeta, \boldsymbol{\sigma})$. Since ζ and ρ are functions of (p, q), we may equivalently use the coordinates

$$\boldsymbol{x} = (p, q, \boldsymbol{\sigma}) \equiv (\gamma, \boldsymbol{\sigma}). \tag{18}$$

This is the *oblate spheroidal coordinate system* in $\mathbb{R}^n$, long known as a useful tool in the theory of special functions [T96]. It is remarkable that this system is so intimately related to the holomorphic extension of the distance function.

Now let ∇ be the gradient with respect to $\boldsymbol{x}$ with $\boldsymbol{y}$ held constant, and let $\Delta = \nabla^2$ be the Laplacian in $\mathbb{R}^n$. Then

$$\gamma^2 = \boldsymbol{z}^2 \ \Rightarrow \ \gamma\nabla\gamma = \boldsymbol{z}, \quad \text{hence} \quad (\nabla\gamma)^2 \equiv \nabla\gamma \cdot \nabla\gamma = 1. \tag{19}$$

Taking the divergence of (19) gives

$$\nabla\gamma \cdot \nabla\gamma + \gamma\Delta\gamma = n, \ \Rightarrow \ \Delta\gamma = \frac{n-1}{\gamma}. \tag{20}$$

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Now

$$\nabla\gamma = \frac{\boldsymbol{z}}{\gamma} = \frac{\bar{\gamma}\,\boldsymbol{z}}{\bar{\gamma}\gamma} \;\Rightarrow\; \nabla p = \frac{p\,\boldsymbol{x} + q\,\boldsymbol{y}}{\bar{\gamma}\,\gamma}, \quad \nabla q = \frac{p\,\boldsymbol{y} - q\,\boldsymbol{x}}{\bar{\gamma}\,\gamma}, \tag{21}$$

therefore

$$\nabla\gamma \cdot \nabla\gamma = 1 \;\Rightarrow\; (\nabla p)^2 - (\nabla q)^2 = 1, \quad \nabla p \cdot \nabla q = 0 \tag{22}$$

$$\nabla\bar{\gamma} \cdot \nabla\gamma = \frac{|\boldsymbol{z}|^2}{\bar{\gamma}\gamma} \;\Rightarrow\; (\nabla p)^2 + (\nabla q)^2 = \frac{r^2 + a^2}{\bar{\gamma}\gamma}, \tag{23}$$

which gives

$$(\nabla p)^2 = \frac{a^2 + p^2}{\bar{\gamma}\gamma}, \qquad (\nabla q)^2 = \frac{a^2 - q^2}{\bar{\gamma}\gamma}. \tag{24}$$

Finally, the real and imaginary parts of (20) give

$$\Delta p = \frac{n-1}{\bar{\gamma}\,\gamma}\,p, \qquad \Delta q = -\frac{n-1}{\bar{\gamma}\,\gamma}\,q\,. \tag{25}$$

We will need to compute volume integrals in the oblate spheroidal coordinates. Note that the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ is [CH62]

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \qquad n \geq 2. \tag{26}$$

Let $d\boldsymbol{\sigma}$ denote the surface measure on the unit sphere S^{n-2} in the hyperplane orthogonal to $\boldsymbol{y} \neq \boldsymbol{0}$, *normalized* so that

$$\int_{S^{n-2}} d\boldsymbol{\sigma} = 1. \tag{27}$$

Then for fixed $\boldsymbol{y}$, the volume measure $d\boldsymbol{x}$ in $\mathbb{R}^n$ is given in cylindrical coordinates by

$$d\boldsymbol{x} = \omega_{n-1}\,\rho^{n-2}d\rho\,d\zeta\,d\boldsymbol{\sigma}. \tag{28}$$

By (8) and (9),

$$\rho^{n-2}d\rho\,d\zeta = \frac{\rho^{n-3}}{2}\,d[\rho^2]\,d\zeta = \frac{\rho^{n-3}}{2a^3}\,d\left[(a^2 + p^2)(a^2 - q^2)\right]d\,[pq]$$
$$= a^{-1}\,\rho^{n-3}(p\,dp - q\,dq)(p\,dq + q\,dp) = a^{-1}\,\rho^{n-3}(p^2 + q^2)\,dp\,dq,$$

where $dp\,dq$ denotes the antisymmetric exterior product of differential forms (see [GS64], for example). Therefore the volume measure in the oblate spheroidal coordinates is given by

$$d\boldsymbol{x} = \frac{\omega_{n-1}}{a}\,\rho^{n-3}\,(p^2 + q^2)\,dp\,dq\,d\boldsymbol{\sigma} = \frac{\omega_{n-1}}{a}\,\rho^{n-3}\,\bar{\gamma}\gamma\,dp\,dq\,d\boldsymbol{\sigma}. \tag{29}$$

3   Holomorphic Potentials and Their Sources

For simplicity, we now assume that $n \geq 3$. The case $n = 2$ is similar but requires some special attention and will be described elsewhere. Consider the *fundamental solution* $\phi(\boldsymbol{x})$ of Laplace's equation defined by

$$\Delta\phi(\boldsymbol{x}) = \delta(\boldsymbol{x}), \quad \lim_{r \to \infty} \phi(\boldsymbol{x}) = 0. \tag{30}$$

It is given uniquely by [CH62]

$$\phi(\boldsymbol{x}) = \frac{1}{\omega_n}\frac{r^{2-n}}{2-n}, \qquad \boldsymbol{x} \in \mathbb{R}^n,\ n \geq 3. \tag{31}$$

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For $n = 3$, $\phi(\boldsymbol{x}) = -1/4\pi r$ is the *Newton-Coulomb potential* with unit mass or charge. Define the *holomorphic potential* in $\mathbb{C}^n$ by

$$\phi(\boldsymbol{z}) = \frac{1}{\omega_n} \frac{\gamma^{2-n}}{2-n}, \qquad \boldsymbol{z} \in \mathbb{C}^n, \; n \geq 3. \tag{32}$$

For odd n, $\phi(\boldsymbol{z})$ inherits the branch cut of $\gamma(\boldsymbol{z})$. For even n, the only singularities occur on $\mathcal{B}$, where $\gamma = 0$. Thus $\gamma(\boldsymbol{z})$ and $\phi(\boldsymbol{z})$ are analytic continuations of $r(\boldsymbol{x})$ and $\phi(\boldsymbol{x})$ to the domains

$$\mathcal{O}_n = \{\boldsymbol{z} \in \mathbb{C}^n \mid p > 0\} = \{\boldsymbol{x} + i\boldsymbol{y} \mid \boldsymbol{x} \notin E_0(\boldsymbol{y})\}, \qquad \text{odd } n \geq 3$$
$$\mathcal{O}_n = \{\boldsymbol{z} \in \mathbb{C}^n \mid \gamma \neq 0\} = \{\boldsymbol{x} + i\boldsymbol{y} \mid \boldsymbol{x} \notin \mathcal{B}(\boldsymbol{y})\}, \qquad \text{even } n \geq 4.$$

Proposition 1 *For fixed $\boldsymbol{y}$, $\phi(\boldsymbol{x} + i\boldsymbol{y})$ is harmonic with respect to $\boldsymbol{x}$ when $\boldsymbol{x} + i\boldsymbol{y} \in \mathcal{O}_n$.*

Proof: By (32), we have

$$\omega_n \nabla \phi(\boldsymbol{z}) = \gamma^{1-n} \nabla \gamma.$$

Thus by (20),

$$\omega_n \Delta \phi(\boldsymbol{z}) = (1-n)\gamma^{-n} + \gamma^{1-n}\Delta\gamma = 0. \quad \blacksquare$$

Our objective is to compute the *source distribution* of $\phi(\boldsymbol{z})$, which we define formally by analogy with (30) as

$$\tilde{\delta}(\boldsymbol{z}) = \Delta \phi(\boldsymbol{z}), \qquad \boldsymbol{z} \in \mathbb{C}^n. \tag{33}$$

This will be shown to be a *generalized function* [GS64] of $\boldsymbol{x}$ for any fixed $\boldsymbol{y}$, meaning that given any sufficienlty smooth "test" function $f(\boldsymbol{x})$, the integral

$$\langle \tilde{\delta}, f \rangle = \int_{\mathbb{R}^n} \tilde{\delta}(\boldsymbol{x} + i\boldsymbol{y}) \, f(\boldsymbol{x}) \, d\boldsymbol{x} \tag{34}$$

defines a bounded linear functional of f. (Generalized functions are more commonly known as *distributions* [Z65]. We use the former term here in order to avoid confusion with the term "source distribution.")

By Proposition 1, $\tilde{\delta}(\boldsymbol{x} + i\boldsymbol{y})$ is supported on $\boldsymbol{x} \in \mathcal{B}$ for even $n \geq 4$, and on $\boldsymbol{x} \in E_0$ otherwise. In any case, it has compact support in the variable $\boldsymbol{x}$, and as $\boldsymbol{y} \to \boldsymbol{0}$, this support contracts to the origin. We will show that

$$\boldsymbol{y} \to \boldsymbol{0} \; \Rightarrow \; \langle \tilde{\delta}, f \rangle \to f(\boldsymbol{0}), \quad \text{hence} \quad \tilde{\delta}(\boldsymbol{x} + i\boldsymbol{y}) \to \delta(\boldsymbol{x}).$$

To compute the generalized function $\tilde{\delta}(\boldsymbol{z})$, we first define the *regularized potential*:

$$\phi_\varepsilon(\boldsymbol{z}) = \theta(p - \varepsilon) \, \phi(\boldsymbol{z}), \quad \varepsilon > 0, \quad \text{where} \quad \theta(\xi) = \begin{cases} 1 & \text{if } \xi > 0 \\ 0 & \text{if } \xi < 0 \end{cases} \tag{35}$$

is the (Heaviside) unit step function. The regularization (35) eliminates the singularities on E_0 and $\mathcal{B}$, replacing them by a discontinuity in ϕ_ε across the spheroid E_ε. Thus, while the source of the *singular potential* $\phi(\boldsymbol{z})$ is concentrated on E_0 or $\mathcal{B}$, that of the regularized potential $\phi_\varepsilon(\boldsymbol{z})$ is concentrated on E_ε. The advantage gained is that while $\phi(\boldsymbol{x} + i\boldsymbol{y})$ is infinite for $\boldsymbol{x} \in \mathcal{B} \subset E_0$, $\phi_\varepsilon(\boldsymbol{x} + i\boldsymbol{y})$ remains bounded in a neighborhood of E_ε.

Now $\phi_\varepsilon(\boldsymbol{z})$ vanishes in the interior of E_ε but is identical to $\phi(\boldsymbol{z})$ in the exterior. Therefore its source distribution, defined as a generalized function by

$$\tilde{\delta}_\varepsilon(\boldsymbol{z}) \equiv \Delta\phi_\varepsilon(\boldsymbol{z}), \tag{36}$$

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represents an *equivalent source distribution* on $\boldsymbol{x} \in E_\varepsilon$ whose potential field simulates that of $\tilde{\delta}(\boldsymbol{z})$ in the exterior of E_ε but vanishes in the interior. We call $\tilde{\delta}_\varepsilon(\boldsymbol{z})$ the *regularized source distribution,* and will define the *singular* source distribution $\tilde{\delta}(\boldsymbol{z})$ as the limit of $\tilde{\delta}_\varepsilon(\boldsymbol{z})$ in the sense of generalized functions, for any fixed $\boldsymbol{y} \in \mathbb{R}^n$. That is,

$$\langle \tilde{\delta}, f \rangle \equiv \lim_{\varepsilon \to 0^+} \langle \tilde{\delta}_\varepsilon, f \rangle \tag{37}$$

for every test function $f(\boldsymbol{x})$ in $\mathbb{R}^n$. Using $\theta'(\xi) = \delta(\xi)$, (35) gives

$$\nabla \phi_\varepsilon = \delta(p - \varepsilon)\, \phi\, \nabla p + \theta(p - \varepsilon) \nabla \phi,$$

hence

$$\Delta \phi_\varepsilon = \delta'(p - \varepsilon)\, \phi\, (\nabla p)^2 + 2\delta(p - \varepsilon)\nabla \phi \cdot \nabla p + \delta(p - \varepsilon)\phi\, \Delta p + \theta(p - \varepsilon)\Delta \phi. \tag{38}$$

By Proposition 1, $\Delta \phi$ vanishes for $p \geq \varepsilon/2$, so the last term in (38) vanishes identically. The remaining terms show that $\tilde{\delta}_\varepsilon(\boldsymbol{x} + i\boldsymbol{y})$ is indeed supported on E_ε as expected. Inserting

$$\nabla \phi = \phi'(\gamma)\nabla \gamma = \frac{\gamma^{1-n}\, \nabla \gamma}{\omega_n}$$

into (38) and using (22)–(25), we have

$$\tilde{\delta}_\varepsilon(\boldsymbol{z}) = [\delta'(p - \varepsilon)\, \phi + 2\delta(p - \varepsilon)\, \phi']\, \frac{a^2 + p^2}{\bar{\gamma}\,\gamma} + \delta(p - \varepsilon)\, \phi\, \frac{n-1}{\bar{\gamma}\,\gamma}\, p. \tag{39}$$

This expression represents a generalized function of $\boldsymbol{x}$. To make sense of it we must apply it to a test function $f(\boldsymbol{x})$ in $\mathbb{R}^n$, assumed to be sufficiently smooth. (That is, f is assumed to possess all derivatives which the ensuing computation requires it to possess. As will be seen, the required degree of smoothness increases with n.) For a given value of $\boldsymbol{y} \in \mathbb{R}^n$, $\tilde{\delta}_\varepsilon$ acts on f as in (34),

$$\langle \tilde{\delta}_\varepsilon, f \rangle = \int_{\mathbb{R}^n} \tilde{\delta}_\varepsilon(\boldsymbol{x} + i\boldsymbol{y})\, f(\boldsymbol{x})\, d\boldsymbol{x}. \tag{40}$$

Using the oblate spheroidal coordinates (18), let us write

$$f(\boldsymbol{x}) = f(\rho\boldsymbol{\sigma} + \zeta\,\hat{\boldsymbol{y}}) = f^\sharp(p, q, \boldsymbol{\sigma}) \equiv f^\sharp(\gamma, \boldsymbol{\sigma}),$$

where the two expressions on the right are obtained by substituting (8) and (9) for ζ and ρ in terms of (p, q). Let

$$\bar{f}^\sharp(\gamma) = \int_{S^{n-2}} f^\sharp(\gamma, \boldsymbol{\sigma})\, d\boldsymbol{\sigma} = \int_{S^{n-2}} f(\rho\boldsymbol{\sigma} + \zeta\,\hat{\boldsymbol{y}})\, d\boldsymbol{\sigma} \equiv \bar{f}(\rho, \zeta). \tag{41}$$

The notations $f^\sharp(\gamma, \boldsymbol{\sigma})$ and $\bar{f}^\sharp(\gamma) \equiv \bar{f}^\sharp(p, q)$ are used for convenience and are *not* meant to imply analyticity in γ. Because of the normalization (27), $\bar{f}^\sharp(\gamma)$ and $\bar{f}(\rho, \zeta)$ are the *means* of $f^\sharp(\gamma, \boldsymbol{\sigma})$ and $f(\rho\boldsymbol{\sigma} + \zeta\,\hat{\boldsymbol{y}})$ over the sphere $S_\gamma = E_p \cap H_q = \{\boldsymbol{x} : |\boldsymbol{x} - \zeta\boldsymbol{\sigma}| = \rho\}$. Using the expression (29) for $d\boldsymbol{x}$, (40) becomes

$$\langle \tilde{\delta}_\varepsilon, f \rangle = \frac{\omega_{n-1}}{a} \int_0^\infty dp \int_{-a}^a dq\, \rho^{n-3}\, \bar{\gamma}\, \gamma\, \tilde{\delta}_\varepsilon(\gamma)\, \bar{f}^\sharp(\gamma),$$

where we have used the fact that $\tilde{\delta}_\varepsilon(\boldsymbol{z})$ in (39) is independent of $\boldsymbol{\sigma}$ to write it as $\tilde{\delta}_\varepsilon(p, q) \equiv \tilde{\delta}_\varepsilon(\gamma)$. By (9),

$$(a^2 + p^2)\partial_p \rho^{n-3} = (n-3)p\, \rho^{n-3}.$$

Hence the first term in (39) gives, upon integrating by parts over p,

$$\int_0^\infty dp \int_{-a}^a dq\, (a^2 + p^2)\rho^{n-3}\delta'(p - \varepsilon)\, \phi(\gamma)\, \bar{f}^\sharp(\gamma)$$

$$= -\int_{-a}^a dq\, \rho^{n-3}\, \left[(n-3)\, \varepsilon\, \phi\, \bar{f}^\sharp + 2\varepsilon\phi\, \bar{f}^\sharp + (a^2 + p^2)(\phi'\, \bar{f}^\sharp + \phi\, \bar{f}^\sharp_p)\right]$$

$$= -\int_{-a}^a dq\, \rho^{n-3}\, \left[(n-1)\, \varepsilon\, \phi\, \bar{f}^\sharp + (a^2 + \varepsilon^2)(\phi'\, \bar{f}^\sharp + \phi\, \bar{f}^\sharp_p)\right],$$

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where $\bar{f}^\sharp_p = \partial_p \bar{f}^\sharp(p, q)$ and the integrand is to be evaluated at $p = \varepsilon$. Inserting the other terms in (39) and simplifying, we obtain

$$\langle \tilde{\delta}_\varepsilon, f \rangle = \frac{a^2 + \varepsilon^2}{a} \, \omega_{n-1} \int_{-a}^{a} dq \, \rho^{n-3} \left[\phi' \bar{f}^\sharp - \phi \bar{f}^\sharp_p \right]. \tag{42}$$

Proposition 2 *For all $n \geq 3$, the regularized source distribution $\tilde{\delta}_\varepsilon(\boldsymbol{x} + i\boldsymbol{y})$ is supported on the oblate spheroid $\boldsymbol{x} \in E_\varepsilon$, and its action on a test function $f(\boldsymbol{x}) = f^\sharp(\gamma, \boldsymbol{\sigma})$ is given by*

$$\langle \tilde{\delta}_\varepsilon, f \rangle = I_\varepsilon(\bar{f}^\sharp), \tag{43}$$

where I_ε is the linear functional defined by

$$I_\varepsilon(\bar{f}^\sharp) = \frac{(a^2 + \varepsilon^2)^{\nu+1}}{a^{n-2} A_n} \int_{-a}^{a} \frac{F^\sharp(\varepsilon + iq)}{(\varepsilon + iq)^{n-1}} \, dq, \quad \nu = \frac{n-3}{2}, \tag{44}$$

with

$$A_n = \frac{\omega_n}{\omega_{n-1}}, \qquad F^\sharp(\gamma) = (a^2 - q^2)^\nu \left[\bar{f}^\sharp(\gamma) + \frac{\gamma \bar{f}^\sharp_p(\gamma)}{n-2} \right]. \tag{45}$$

Proof: This follows immediately from (42), using

$$\rho^2 = \frac{(a^2 + \varepsilon^2)(a^2 - q^2)}{a^2}. \quad \blacksquare$$

Let us verify that as $\boldsymbol{y} \to \boldsymbol{0}$ and the source disk shrinks to a point, the source of the singular potential ϕ contracts to the usual point source. Letting $q = a\xi$ in (45), we have

$$F^\sharp(\gamma) = a^{n-3}(1 - \xi^2)^\nu \left[\bar{f}^\sharp(\gamma) + \frac{\gamma \bar{f}^\sharp_p(\gamma)}{n-2} \right], \quad \text{where} \quad \gamma = \varepsilon + ia\xi.$$

Therefore

$$I_\varepsilon(\bar{f}^\sharp) = \frac{(a^2 + \varepsilon^2)^{\frac{n-1}{2}}}{A_n} \int_{-1}^{1} \frac{(1 - \xi^2)^\nu}{(\varepsilon + ia\xi)^{n-1}} \left[\bar{f}^\sharp(\gamma) + \frac{\gamma \bar{f}^\sharp_p(\gamma)}{n-2} \right] d\xi$$

and

$$\lim_{a \to 0} I_\varepsilon(\bar{f}^\sharp) = \left[\bar{f}^\sharp(\varepsilon) + \frac{\varepsilon \bar{f}^\sharp_p(\varepsilon)}{n-2} \right] \frac{K_n}{A_n},$$

where

$$K_n = \int_{-1}^{1} (1 - \xi^2)^\nu \, d\xi = B\left(\tfrac{1}{2}, \tfrac{n-1}{2}\right) = \frac{\sqrt{\pi} \, \Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} = \frac{\omega_n}{\omega_{n-1}} = A_n.$$

Thus by (43),

$$\lim_{a \to 0} \langle \tilde{\delta}_\varepsilon, f \rangle = \bar{f}^\sharp(\varepsilon) + \frac{\varepsilon \bar{f}^\sharp_p(\varepsilon)}{n-2}.$$

Now let $\varepsilon \to 0$, and note that since E_ε contracts to the origin,

$$\lim_{\varepsilon \to 0} \bar{f}^\sharp(\varepsilon) = \lim_{\varepsilon \to 0} \int_{S^{n-2}} f^\sharp(\varepsilon, \boldsymbol{\sigma}) d\boldsymbol{\sigma} = f(\boldsymbol{0}), \quad \text{hence} \quad \lim_{\varepsilon \to 0} \lim_{a \to 0} \langle \tilde{\delta}_\varepsilon, f \rangle = f(\boldsymbol{0}).$$

If we assume that the two limits can be exchanged (as will be verified later) and use the definition (37) of $\tilde{\delta}$, this states that $\lim_{a \to 0} \langle \tilde{\delta}, f \rangle = f(\boldsymbol{0})$, giving the following important result.

Theorem 1 *The singular source distribution $\tilde{\delta}(\boldsymbol{z})$ is an extension of the usual point-source distribution in $\mathbb{R}^n$, in the sense that*

$$\boldsymbol{y} \to \boldsymbol{0} \implies \tilde{\delta}(\boldsymbol{x} + i\boldsymbol{y}) \to \delta(\boldsymbol{x}). \quad \blacksquare \tag{46}$$

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4 Singular Source Distributions

We are ready at last compute the singular source distributions. By (37) and (43), they given by the limit

$$\langle\,\tilde{\delta}, f\,\rangle = \lim_{\varepsilon\to 0} I_\varepsilon(\bar{f}^\sharp). \tag{47}$$

However, we cannot simply let $\varepsilon = 0$ in the expression (44) for $I_\varepsilon(\bar{f}^\sharp)$, since the resulting integral diverges. Recall the decomposition of E_0 given in Equation (14):

$$E_0 = E_0^+ \cup E_0^- \cup \mathcal{B}, \tag{48}$$

where $E_0^\pm$ are the interiors of the front and back sides of E_0 and $\mathcal{B}$ is its rim, the branching sphere. This decomposition recognizes the nature of E_0 as a limit of closed ellipsoids. Although the two open disks $E_0^\pm$ look identical to a *continuous* function, they look distinct to a generalized function like $\tilde{\delta}$ which is singular across E_0. Furthermore, the oblate spheroidal coordinates are an ideal tool for resolving this decomposition since $q > 0$ on E_0^+, $q < 0$ on E_0^- and $q = 0$ on $\mathcal{B}$. We will compute $\langle\,\tilde{\delta}, f\,\rangle$ by decomposing the integral $I_\varepsilon(\bar{f}^\sharp)$ in a way similar to (48) and then taking the limit $\varepsilon \to 0$. The integral over $E_0^+ \cup E_0^-$ gives a sum of *single and double layer distibutions* of dimension $n-1$ over the interior of the disk E_0, while the integral over $\mathcal{B}$ gives a *boundary distribution* of dimension $n-2$ over $\mathcal{B}$. All these distributions are well-defined, giving a finite expression for $\langle\,\tilde{\delta}, f\,\rangle$.

The divergence of $I_0(\bar{f}^\sharp)$ is therefore caused by the attempt to represent the boundary distribution on $\mathcal{B}$ as part of an integral of dimension $n-1$ over E_0, and the above regularization simply amounts to recognizing this fact.

To regularize the integral

$$I_\varepsilon(\bar{f}^\sharp) = \frac{(a^2 + \varepsilon^2)^{\nu+1}}{a^{n-2}A_n} \int_{-a}^{a} \frac{F^\sharp(\varepsilon + iq)}{(\varepsilon + iq)^{n-1}}\, dq, \tag{49}$$

define the Taylor coefficients

$$T_m(\varepsilon) = \frac{1}{m!}\partial_q^m F^\sharp(\varepsilon + iq)\,\big|_{q=0}, \qquad T_m \equiv T_m(0) \tag{50}$$

and the Taylor polynomials approximating $F^\sharp(\varepsilon + iq)$ and $F^\sharp(iq)$ to order q^{n-2},

$$F_{n-2}^\sharp(\varepsilon, q) = \sum_{m=0}^{n-2} q^m T_m(\varepsilon), \quad F_{n-2}^\sharp(q) = \sum_{m=0}^{n-2} q^m T_m\,. \tag{51}$$

We rewrite the integral in (49) as

$$\int_{-a}^{a} \frac{F^\sharp(\varepsilon + iq)}{(\varepsilon + iq)^{n-1}}\, dq = \int_{-a}^{a} \frac{F^\sharp(\varepsilon + iq) - F_{n-2}^\sharp(\varepsilon, q)}{(\varepsilon + iq)^{n-1}}\, dq + \sum_{m=0}^{n-2} i^{-m}T_m(\varepsilon)\lambda_k^m(\varepsilon), \tag{52}$$

where

$$\lambda_k^m(\varepsilon) = \int_{-a}^{a} \frac{i^m q^m dq}{(\varepsilon + iq)^k}, \quad 0 \le m < k. \tag{53}$$

If $f(\boldsymbol{x})$ is differentiable to order $n - 2$, so is $F^\sharp(\varepsilon + iq) \equiv F^\sharp(\varepsilon, q)$. Then

$$F^\sharp(\varepsilon + iq) - F_{n-2}^\sharp(\varepsilon, q) = O(q^{n-1})$$

and the integral on the right-hand side of (52) has a finite limit as $\varepsilon \to 0$. It therefore remains to compute the limit of the sum. We begin by finding

$$\lambda_1^0(\varepsilon) = \int_{-a}^{a} \frac{dq}{\varepsilon + iq} = 2\varepsilon\int_0^a \frac{dq}{\varepsilon^2 + q^2} = 2\tan^{-1}\frac{a}{\varepsilon} = \pi - 2\tan^{-1}\frac{\varepsilon}{a} \equiv \alpha(\varepsilon). \tag{54}$$

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All the other integrals (53) can be computed from the recursion relations

$$\lambda_{k+1}^0(\varepsilon) = -\frac{1}{k}\partial_\varepsilon \lambda_k^0(\varepsilon) = \frac{(-1)^k}{k!}\partial_\varepsilon^k \alpha(\varepsilon)$$

$$\lambda_k^{m+1}(\varepsilon) = i^m \int_{-a}^a \frac{(\varepsilon + iq - \varepsilon)q^m dq}{(\varepsilon + iq)^k} = \lambda_{k-1}^m(\varepsilon) - \varepsilon \lambda_k^m(\varepsilon).$$

We are interested in the limit $\varepsilon \to 0$, where these relations imply

$$\lambda_{k+1} \equiv \lambda_{k+1}^0(0) = \frac{(-1)^k}{k!}\partial_\varepsilon^k \alpha(0) \equiv (-1)^k L_k \tag{55}$$

$$\lambda_k^m(0) = \lambda_{k-m}^0(0) = \lambda_{k-m}, \quad 0 \le m < k. \tag{56}$$

The Taylor coefficients L_k of $\alpha(\varepsilon)$ are obtained from the expansion

$$\alpha(\varepsilon) = \pi - 2\tan^{-1}\frac{\varepsilon}{a} = \pi + 2\sum_{l=1}^\infty \frac{(-1)^l}{2l-1}\frac{\varepsilon^{2l-1}}{a^{2l-1}}. \tag{57}$$

Thus

$$\lambda_1 = \pi, \quad \lambda_{2l} = \frac{2(-1)^{l+1}}{(2l-1)a^{2l-1}}, \quad \lambda_{2l+1} = 0, \quad l \ge 1. \tag{58}$$

By (56), this gives finite values for all the coefficients $\lambda_k^m(0)$. Note that the original expression (53) diverges if we set $\varepsilon = 0$. The finiteness of the limits depends on delicate cancellations of contributions from positive and negative values of q when $\varepsilon > 0$, just as happens in Cauchy's principal value integral. In fact, (54) shows that *the present regularization reduces to the principal value integral when $k = 1$.*

Using (52), we can now compute the limit $\varepsilon \to 0$ in (47):

$$\langle \tilde\delta, f \rangle = V_n(f) + \frac{a}{A_n}\sum_{m=0}^{n-2} i^{-m} T_m \lambda_{n-m-1}, \tag{59}$$

$$V_n(f) \equiv \frac{i^{1-n}a}{A_n} \int_{-a}^a \frac{F^\sharp(iq) - F_{n-2}^\sharp(q)}{q^{n-1}}\, dq. \tag{60}$$

$V_n(f)$ will be shown to be a *bounded linear functional* of f, and this establishes $\tilde\delta$ as a well-defined generalized function.

Since $\bar f^\sharp(0)$ is the mean of f over $\mathcal{B}$, the terms in the sum in (59) represent means of f and its derivatives over $\mathcal{B}$. On the other hand, (60) represents an integral of f and its normal derivative over the *interior $E_0^+ \cup E_0^-$*, since the boundary terms have already been subtracted in the form of $F_{n-2}^\sharp(q)$. Thus, (59) is the promised decomposition of the source distribution corresponding to (48).

Equation (59) can be greatly simplified because of certain symmetries satisfied by $\bar f^\sharp(iq)$. We claim that

$$\bar f^\sharp(-iq) = \bar f^\sharp(iq), \quad \bar f_p^\sharp(-iq) = -\bar f_p^\sharp(iq), \quad \bar f_q^\sharp(-iq) = -\bar f_q^\sharp(iq). \tag{61}$$

To see this, note that the coordinates $\gamma = \pm iq$ denote *the same point* of E_0, regarded as belonging to $E_0^\pm$. Since the test function $f(\boldsymbol{x})$ is continuous across E_0, so is its integral $\bar f^\sharp(iq)$, and this proves the first relation. On the other hand, p increases in the $\pm\zeta$ direction on $E_0^\pm$, which proves the second relation. Finally, the third relation follows from the first by differentiation.

By (61), the function

$$F^\sharp(iq) = (a^2 - q^2)^\nu \left[\bar f^\sharp(iq) + i\frac{q\bar f_p^\sharp(iq)}{n-2}\right] \tag{62}$$

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is *even,* so its odd Taylor coefficients vanish:

$$m \text{ is odd} \quad \Rightarrow \quad T_m = 0. \tag{63}$$

Furthermore, (60) shows that

$$n \text{ is even} \quad \Rightarrow \quad V_n(f) = 0. \tag{64}$$

Considering the cases of odd and even n separately and inserting the values of λ_k from (58) gives the following result.

Theorem 2 *The singular source distribution $\tilde{\delta}(\boldsymbol{z})$ is a bounded linear functional which acts on a test function $f \in C^{n-2}(\mathbb{R}^n)$ as follows. For even $n = 2k + 2 \geq 4$,*

$$\langle \tilde{\delta}, f \rangle = \frac{\pi a}{A_n} (-1)^k T_{2k}, \tag{65}$$

where A_n and T_{2k} are given by (45) and (50). For odd $n = 2k + 3 \geq 3$,

$$\langle \tilde{\delta}, f \rangle = V_n(f) + \frac{2(-1)^k}{A_n} \sum_{l=0}^{k} \frac{a^{2l-2k} T_{2l}}{2k - 2l + 1}, \tag{66}$$

where $V_n(f)$ is given by (60).

Proof: To prove (65), note first that $V_n(f)$ vanishes by (64). Furthermore, the sum in (59) reduces to a single term because $T_m \lambda_{n-m-1} = 0$ unless m is even, which implies that $n - m - 1$ is odd; but the only nonvanishing coefficient λ_l with odd l is $\lambda_1 = \pi$.

Equation (66) follows directly from (59), (58) and (63). To establish $\tilde{\delta}$ as a distribution, we must still prove that $V_n(f)$ is a bounded linear functional for odd n. Since the integrand in (60) is even when n is odd, we have

$$V_n(f) = \frac{2i^{1-n}a}{A_n} \int_0^a \frac{F^\sharp(iq) - F^\sharp_{n-2}(q)}{q^{n-1}} \, dq.$$

By Taylor's theorem,

$$F^\sharp(iq) - F^\sharp_{n-2}(q) = \frac{1}{(n-2)!} \int_0^q (q-w)^{n-2} (F^\sharp)^{(n-1)}(iw) \, dw.$$

Therefore

$$|F^\sharp(iq) - F^\sharp_{n-2}(q)| \leq q^{n-1} M, \qquad M = \frac{\max |(F^\sharp)^{(n-1)}(iw)|}{(n-2)!},$$

and

$$|V_n(f)| \leq 2a^2 M.$$

Since M depends boundedly on f, this shows that $V_n(f)$ is a bounded linear functional as claimed. ∎

Thus, for even $n \geq 4$, the integral over the interior of E_0 vanishes and the singular source is concentrated on $\mathcal{B}$. This is to be expected, since γ^{2-n} has no branch cut and therefore the only singularity occurs on the boundary.

We can now derive a useful expression for the source distribution $\tilde{\delta}(\boldsymbol{x} + i\boldsymbol{y})$ in the cylindrical coordinates $(\rho, \zeta, \boldsymbol{\sigma})$ adapted to $\boldsymbol{y} \neq \boldsymbol{0}$. As a byproduct, it will be seen that for all $n \geq 3$, the test function $f(\boldsymbol{x})$ need only be in $C^k(\mathbb{R}^n)$ with $k = [\frac{n-1}{2}]$, instead of $C^{n-2}(\mathbb{R}^n)$ as assumed in Theorem 2.

Our first task is to rewrite the function $F^\sharp(iq)$ in (62) in terms of cylindrical coordinates. Recall that

$$\zeta = \frac{pq}{a}, \qquad \rho = \frac{\sqrt{(a^2 + p^2)(a^2 - q^2)}}{a} \tag{67}$$

$$\boldsymbol{x} = \rho\boldsymbol{\sigma} + \zeta\,\hat{\boldsymbol{y}}, \quad \boldsymbol{\sigma} \in S^{n-2} \perp \boldsymbol{y},$$

$$f(\rho\boldsymbol{\sigma} + \zeta\,\hat{\boldsymbol{y}}) = f^\sharp(p, q, \boldsymbol{\sigma}). \tag{68}$$

Then the mean of f on $S_\gamma = E_p \cap H_q$ becomes

$$\bar{f}^\sharp(p, q) = \int_{S^{n-2}} h(\rho\boldsymbol{\sigma} + \zeta\,\hat{\boldsymbol{y}})\, d\boldsymbol{\sigma} \equiv \bar{f}(\rho, \zeta), \tag{69}$$

which is the mean of f on the sphere $S_{\rho, \zeta} = \{\boldsymbol{x} : |\boldsymbol{x} - \zeta\,\hat{\boldsymbol{y}}| = \rho\}$.

It will be shown elsewhere [K00] that the source distributions for odd n can be derived from those for even n by integrating over one of the coordinates. Specifically, we have:

Theorem 3 *Let $n \geq 3$, and denote points in $\mathbb{C}^{n+1}$ by*

$$\mathbf{z} = \mathbf{x} + i\mathbf{y} = (\boldsymbol{z}, s + it), \qquad \boldsymbol{z} \in \mathbb{C}^n.$$

For fixed $\mathbf{y} \neq \mathbf{0}$, the singular source distributions $\tilde{\delta}_n(\boldsymbol{x} + i\boldsymbol{y})$ in $\mathbb{R}^n$ and $\tilde{\delta}_{n+1}(\mathbf{x} + i\mathbf{y})$ in $\mathbb{R}^{n+1}$ are related by

$$\tilde{\delta}_n(\boldsymbol{z}) = \int_{-\infty}^{\infty} ds\, \tilde{\delta}_{n+1}(\boldsymbol{z}, s). \tag{70}$$

That is, for a test function $f(\boldsymbol{x})$ in $\mathbb{R}^n$ we have

$$\langle\, \tilde{\delta}_n\,, f\,\rangle = \int_{-\infty}^{\infty} ds \int_{\mathbb{R}^n} d\boldsymbol{x}\ \tilde{\delta}_{n+1}(\boldsymbol{x} + i\boldsymbol{y}, s)f(\boldsymbol{x}). \tag{71}$$

For $\boldsymbol{y} = \mathbf{0}$, (70) follows from $\delta_{n+1}(\boldsymbol{x}, s) = \delta_n(\boldsymbol{x})\,\delta(s)$. For $\boldsymbol{y} \neq \mathbf{0}$, this tensor product decomposition fails but Theorem 3 still holds. We therefore state the following theorem for the simpler case of *even $n \geq 4$*, though the transformation to cylindrical coordinates derived below will be used later in the explicit computation of the extended source in $\mathbb{R}^3$.

Theorem 4 *For $n = 2k + 2 \geq 4$, the source distribution $\tilde{\delta}$ is a bounded linear functional on test fucntions $f(\boldsymbol{x})$ in $C^k(\mathbb{R}^n)$ whose action is given in the cylindrical coordinates $(\rho, \zeta, \boldsymbol{\sigma})$ by*

$$\langle\, \tilde{\delta}, f\,\rangle = \frac{a\sqrt{\pi}}{\Gamma(k + \frac{1}{2})}\, D_\rho^k\, F(\rho)\, \Big|_{\rho = a}, \tag{72}$$

where

$$D_\rho = \frac{\partial}{\partial(\rho^2)} = \frac{1}{2\rho}\frac{\partial}{\partial\rho}, \quad F(\rho) = \rho^{2k-1}\left[\bar{f}(\rho, 0) + i\frac{a^2 - \rho^2}{2ka}\,\bar{f}_\zeta(\rho, 0)\right]. \tag{73}$$

Proof: By (67), we have

$$\bar{f}_p^\sharp(p, q) = \frac{\rho p}{a^2 + p^2}\,\bar{f}_\rho(\rho, \zeta) + \frac{q}{a}\,\bar{f}_\zeta(\rho, \zeta). \tag{74}$$

Since

$$p = 0 \;\Rightarrow\; q^2 = a^2 - \rho^2,$$

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it follows from (62) that

$$F(iq) = (a^2 - q^2)^\nu \left[\bar{f}^\sharp(0, q) + i \frac{q \bar{f}_p^\sharp(0, q)}{n - 2} \right]$$

$$= \rho^{n-3} \left[\bar{f}(\rho, 0) + i \frac{q^2}{(n-2)a} \bar{f}_\zeta(\rho, 0) \right] = F(\rho). \tag{75}$$

Note that (75) makes sense only because $F(iq)$ is *even* and so does not depend on the sign of q, which cannot be recovered from ρ when $\zeta = 0$ since $\operatorname{sgn} q = \operatorname{sgn} \zeta$. With

$$D_q = \frac{\partial}{\partial(q^2)} = \frac{1}{2q} \frac{\partial}{\partial q} = -D_\rho \quad \text{when} \quad p = 0,$$

we have

$$F(iq) = \sum_{m=0}^\infty T_{2m} q^{2m} \;\Rightarrow\; T_{2m} = \frac{1}{m!} D_q^m F(iq) \Big|_{q=0} = \frac{(-1)^m}{m!} D_\rho^m F(\rho) \Big|_{\rho=a}.$$

For $n = 2k + 2 \geq 4$,

$$\omega_n = \frac{2\pi^{k+1}}{\Gamma(k+1)} = \frac{2\pi^{k+1}}{k!}, \quad \omega_{n-1} = \frac{2\pi^{k+\frac{1}{2}}}{\Gamma(k+\frac{1}{2})},$$

and (65) gives

$$\langle \tilde{\delta}, f \rangle = \frac{\pi a \omega_{n-1}}{\omega_n} (-1)^k T_{2k} = \frac{a\sqrt{\pi}}{\Gamma(k+\frac{1}{2})} D_\rho^k F(\rho) \Big|_{\rho=a}.$$

Now $F(\rho)$ already contains one derivative ($\bar{f}_\zeta$) and D_ρ^k computes k more, hence it suffices to have $f \in C^{k+1}(\mathbb{R}^n)$. But the highest derivative of f occuring in (72) is

$$\frac{\rho^{2k-1}}{(2\rho)^k} \cdot i \frac{a^2 - \rho^2}{2ka} \partial_\rho^k \bar{f}_\zeta(\rho, 0),$$

which vanishes at $\rho = a$. Hence the highest nonvanishing derivative is of order k, so it suffices for $\bar{f}(\rho, \zeta)$ to be k times continuously differentiable as claimed, and so does $f(\boldsymbol{x})$. ∎

An application of (71) now shows that for $n = 2k + 1$, we need $f \in C^k(\mathbb{R}^n)$ in order for $\langle \tilde{\delta}_n, f \rangle$ to make sense. Thus for *any* $n \geq 3$, f needs to be in $C^k(\mathbb{R}^n)$ with

$$k = \left[\frac{n-1}{2} \right] = \begin{cases} \frac{n-2}{2}, & n \text{ even} \\ \frac{n-1}{2}, & n \text{ odd}. \end{cases}$$

5   Computation of Sources in $\mathbb{R}^3$ and $\mathbb{R}^4$

We now compute the singular source distribution in $\mathbb{R}^3$ explicitly and interpret the result. Equation (62) becomes

$$F^\sharp(iq) = \bar{f}^\sharp(iq) + iq\bar{f}_p^\sharp(iq), \tag{76}$$

hence $T_0 = \bar{f}^\sharp(0)$, $T_1 = 0$, and (51) becomes

$$F_1^\sharp(q) = \bar{f}^\sharp(0).$$

Equation (66) therefore gives

$$\langle \tilde{\delta}, f \rangle = -a \int_{-a}^a \frac{\bar{f}^\sharp(iq) + iq\bar{f}_p^\sharp(iq) - \bar{f}^\sharp(0)}{q^2} \, dq + \bar{f}^\sharp(0).$$

Using the symmetry of the integrand, we obtain the following result.

Proposition 3 *The singular source distribution*

$$\tilde{\delta}(\boldsymbol{x} + i\boldsymbol{y}) = -\Delta \frac{1}{4\pi\gamma}, \quad \boldsymbol{x} + i\boldsymbol{y} \in \mathbb{C}^3,$$

is a bounded linear functional whose action on a test function $f(\boldsymbol{x}) = f^\sharp(\gamma, \boldsymbol{\sigma})$ in $C^1(\mathbb{R}^3)$, is given in oblate spheroidal and cylindrical coordinates by

$$\langle \tilde{\delta}, h \rangle = L_0 + L_1 + iL_2, \tag{77}$$

where

$$L_0 = \bar{f}^\sharp(0) = \bar{f}(a, 0)$$

$$L_1 = -a \int_0^a \frac{\bar{f}^\sharp(iq) - \bar{f}^\sharp(0)}{q^2} \, dq = -a \int_0^a \frac{\bar{f}(\rho, 0) - \bar{f}(a, 0)}{(a^2 - \rho^2)^{3/2}} \, \rho \, d\rho$$

$$L_2 = -a \int_0^a \frac{\bar{f}_p^\sharp(iq)}{q} \, dq = -\int_0^a \frac{\bar{f}_\zeta(\rho, 0)}{\sqrt{a^2 - \rho^2}} \, \rho \, d\rho.$$

Proof: The action in oblate spheroidal coordinates follows immediately from (66). To obtain the action in cylindrical coordinates, use (74). ∎

Note that since

$$\lim_{a \to 0} \bar{f}(a, 0) = f(\boldsymbol{0}) \quad \text{and} \quad \lim_{a \to 0} L_1 = \lim_{a \to 0} L_2 = 0,$$

(77) shows that

$$\boldsymbol{y} \to \boldsymbol{0} \implies \tilde{\delta}(\boldsymbol{x} + i\boldsymbol{y}) \to \delta(\boldsymbol{x}).$$

This was already seen in Theorem 1 for all $n \geq 3$, but that proof was less rigorous because it depended on the assumption that the order of the limits $a \to 0$ and $\varepsilon \to 0$ can be exchanged.

We now state some other interesting properties of the expression (77) which will help interpret its three terms.

Proposition 4 *The monopole and dipole moments of $\tilde{\delta}(\boldsymbol{x} + i\boldsymbol{y})$ in $\mathbb{R}^3$ are*

$$Q \equiv \int_{\mathbb{R}^3} \tilde{\delta}(\boldsymbol{x} + i\boldsymbol{y}) \, d\boldsymbol{x} = 1, \qquad \boldsymbol{P} \equiv \int_{\mathbb{R}^3} \boldsymbol{x} \, \tilde{\delta}(\boldsymbol{x} + i\boldsymbol{y}) \, d\boldsymbol{x} = -i\boldsymbol{y}.$$

Given a point source with general *complex coordinates $\boldsymbol{z}_s = \boldsymbol{x}_s + i\boldsymbol{y}_s \in \mathbb{C}^3$, the* centroid *of its charge distribution is*

$$\boldsymbol{C}(\boldsymbol{z}_s) \equiv \int_{\mathbb{R}^3} \boldsymbol{x} \, \tilde{\delta}(\boldsymbol{x} - \boldsymbol{z}_s) \, d\boldsymbol{x} = \boldsymbol{z}_s. \tag{78}$$

Proof: To compute Q, apply Proposition 3 with $f(\boldsymbol{x}) \equiv 1$. To find $\boldsymbol{P}$, apply it to the *vector-valued* test function

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x} = \rho\boldsymbol{\sigma} + \zeta\hat{\boldsymbol{y}}.$$

Finally,

$$\boldsymbol{C}(\boldsymbol{z}_s) = \boldsymbol{x}_s \int_{\mathbb{R}^3} \tilde{\delta}(\boldsymbol{x} - \boldsymbol{x}_s - i\boldsymbol{y}_s) \, d\boldsymbol{x} + \int_{\mathbb{R}^3} (\boldsymbol{x} - \boldsymbol{x}_s) \, \tilde{\delta}(\boldsymbol{x} - \boldsymbol{x}_s - i\boldsymbol{y}_s) \, d\boldsymbol{x} = \boldsymbol{x}_s + i\boldsymbol{y}_s = \boldsymbol{z}_s. \quad \blacksquare$$

Equation (78) is a natural extension to $\mathbb{C}^3$ of the formula

$$\int_{\mathbb{R}^3} \boldsymbol{x} \, \delta(\boldsymbol{x} - \boldsymbol{x}_s) \, d\boldsymbol{x} = \boldsymbol{x}_s, \qquad \boldsymbol{x}_s \in \mathbb{R}^3.$$

Proposition 4 sheds some light on the nature of the source distribution $\tilde{\delta}$.

- L_0 is the mean of f over the rim $\mathcal{B}$. Since $L_1 = L_2 = 0$ when $f(\boldsymbol{x}) \equiv 1$, we see that the "charge" Q resides entirely on this rim.

- L_1 is an integral of f over E_0 which does not involve its derivatives, thus representing a *single layer distribution* on E_0. Actually, since the contributions from the rim are subtracted, the single layer resides on the *interior* of E_0.

- L_2 is an integral of the normal derivative of f over E_0, so it represents a *double layer distribution* which may be regarded as equal and opposite charge distributions on E_0^+ and E_0^-. This is confirmed by $\boldsymbol{P} = -i\boldsymbol{y}$.

Proposition 5 *The source distribution in $\mathbb{C}^4$,*

$$\tilde{\delta}(\boldsymbol{z}) = -\Delta \frac{1}{4\pi^2 \gamma^2} \, ,$$

acts on a test function $f \in C^1(\mathbb{R}^4)$ in cylindrical coordinates as follows:

$$\langle \, \tilde{\delta}, f \, \rangle = \bar{f}(a, 0) + a\bar{f}_\rho(a, 0) - ia\bar{f}_\zeta(a, 0). \tag{79}$$

Proof: From (73) we find

$$F(\rho) = \rho \left[\bar{f}(\rho, 0) + i \, \frac{a^2 - \rho^2}{2a} \, \bar{f}_\zeta(\rho, 0) \right],$$

and (72) becomes $\langle \, \tilde{\delta}, f \, \rangle = \partial_\rho F(a)$, which gives (79). ∎

As $a \to 0$, this gives $\langle \, \tilde{\delta}, f \, \rangle \to f(\boldsymbol{0})$, which confirms that $\tilde{\delta}(\boldsymbol{x} + i\boldsymbol{y}) \to \delta(\boldsymbol{x})$. This property and $Q = 1, \boldsymbol{P} = -i\boldsymbol{y}, \boldsymbol{C}(\boldsymbol{z}_S) = \boldsymbol{z}_S$, as in Proposition 4, can be shown to hold for *all* n [K00].

The holomorphic potential in $\mathbb{C}^4$ was derived from a different point of view in [K94, Section 11.2], where it was shown to decompose into *causal* (retarded) and *anticausal* (advanced) "physical wavelets." These are closely related to the *complex-source pulsed beams* in the engineering literature [HF89].

6 Connection to Spacetime and Wave Equations

It was proposed in Section 1 that point sources in physics be replaced by extended sources based on a continuation of physics to complex spacetime. The distribution $\tilde{\delta}(\boldsymbol{z})$ in complex space ($\mathbb{C}^n$) seems like a promising model, since it is supported on the disk E_0 (when n is odd) or the sphere $\mathcal{B}$ (when n is even). But to do physics we need to add *time* to this complex-space formalism. We will first give an argument suggesting that time is *already* included in $\mathbb{C}^{n+1}$, then justify this by proving that with this interpretation, the extended source distribution $\tilde{\delta}(\mathbf{z})$ in $\mathbb{C}^{n+1}$ acts as a *propagator* in spacetime, generating solutions of the Cauchy problem for the wave equation from their initial values.

Let us write

$$\mathbf{z} = \mathbf{x} + i\mathbf{y} = (\boldsymbol{z}, s + it) \in \mathbb{C}^{n+1}, \qquad\qquad \boldsymbol{z} \in \mathbb{C}^n$$
$$\mathbf{x} = (\boldsymbol{x}, s), \quad \mathbf{y} = (\boldsymbol{y}, t) \in \mathbb{R}^{n+1}, \qquad\qquad |\boldsymbol{x}| = r.$$

Analytic potential theory in $\mathbb{C}^{n+1}$ is based on the complex distance function $\gamma(\mathbf{z})$ defined by

$$\gamma^2 \equiv \mathbf{z}^2 = \boldsymbol{z}^2 + (s + it)^2, \qquad \text{Re } \gamma \geq 0,$$

which reduces to the *Euclidean* metric in $\mathbb{R}^{n+1}$ when $\mathbf{y} \to \boldsymbol{0}$,

$$\gamma^2 \to \mathbf{x}^2 = r^2 + s^2,$$

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and to the *Lorentzian* metric in $\mathbb{R}^{n,1}$ when $s \to 0$ and $\boldsymbol{y} \to \boldsymbol{0}$,

$$\gamma^2 \to r^2 - t^2.$$

This suggests that *the imaginary part t of the complex space coordinate $z_{n+1} = s + it$ should be interpreted as time.* For this reason, we refer to its real part s as the *Euclidean time.*

The idea of time as an imaginary space coordinate dates back to Minkowski [M23], who realized in 1908 that Einstein's new special relativity theory can be based on a unified four-dimensional spacetime. More recently, complex spacetime has become an important concept in quantum field theory [SW64, GJ87], twistor theory [PR86] and string theory [P98]. Even so, it is generally regarded as a useful mathematical tool rather than a fundamental aspect of physical reality. For example, see the discussion in [MTW73, p. 51].

In previous work [K77, K78, K87, K90, K94], I have attempted to make complex spacetime "concrete" by giving detailed physical interpretations of the imaginary as well as the real coordinates. This is also the present motivation for developing holomorphic potential theory, where the gemetrical and physical significance of $\boldsymbol{y}$ has been emphasized. (The *physical* significance of $\boldsymbol{y}$ is related to the directivity of the physical wavelets associated with $\phi(\boldsymbol{x} + i\boldsymbol{y})$ [HF89, K94, K00].)

Assume for simplicity that the Euclidean world is completely democratic with respect to the real space variables $\boldsymbol{x} \in \mathbb{R}^n$ and the Euclidean time s, so that there is no prefered direction in $\mathbb{R}^{n+1}$. Then if $\mathbf{y} \neq \boldsymbol{0}$, we may choose a coordinate system in which

$$\mathbf{y} = (\boldsymbol{0}, t), \quad t = |\mathbf{y}| > 0, \qquad \mathbf{z} = (\boldsymbol{x}, s + it) \in \mathbb{C}^{n+1}.$$

In that case, the cylindrical coordinates in $\mathbb{R}^{n+1}$ are

$$\zeta = \mathbf{x} \cdot \hat{\mathbf{y}} = s, \qquad \rho = \sqrt{\mathbf{x}^2 - \zeta^2} = r,$$

and a vector $\mathbf{x} \in \mathbb{R}^{n+1}$ is represented by

$$\mathbf{x} = r\boldsymbol{\sigma} + s\hat{\mathbf{y}}, \qquad \boldsymbol{\sigma} \in S^{n-1} \subset \mathbb{R}^n,$$

where we do not distinguish between $\boldsymbol{\sigma} \in \mathbb{R}^n$ and $(\boldsymbol{\sigma}, 0) \in \mathbb{R}^{n+1}$. As before, we denote by $d\boldsymbol{\sigma}$ the normalized surface measure on S^{n-1}, so that $\int d\boldsymbol{\sigma} = 1$ and

$$\bar{f}(r, s) \equiv \int_{S^{n-1}} f(r\boldsymbol{\sigma} + s\hat{\mathbf{y}}) \, d\boldsymbol{\sigma}$$

is the mean of $f(\boldsymbol{x}, s)$ over the sphere $|\boldsymbol{x}| = r$.

For simplicity, we assume to begin with that the dimension n of space is *odd*, so that $n + 1$ is even and (72) can be applied to $\tilde{\delta}(\mathbf{z})$. We will later extend our results to *even* values of n by applying (71). Using (72) with the substitutions

$$n \to n + 1 \equiv 2k + 2, \quad \rho \to r, \quad \zeta \to s, \quad a \to t \tag{80}$$

gives the action of $\tilde{\delta}(\mathbf{x} + i\mathbf{y})$ on a test function $f(\mathbf{x})$:

$$\langle \tilde{\delta}, f \rangle = \frac{t\sqrt{\pi}}{\Gamma(k + \frac{1}{2})} \, D_r^k F(r) \, \Big|_{r=t}, \tag{81}$$

where

$$D_r = \frac{1}{2r} \frac{\partial}{\partial r}, \quad F(r) = r^{2k-1} \left[\bar{f}(r, 0) + i \frac{t^2 - r^2}{2kt} \bar{f}_s(r, 0) \right]. \tag{82}$$

To establish a connection with the wave equation, define the function $\tilde{f}(\mathbf{z})$ on $\mathbb{C}^{n+1}$ by the convolution

$$\tilde{f}(\mathbf{z}) \equiv \int_{\mathbb{R}^{n+1}} \tilde{\delta}(\mathbf{z} - \mathbf{x}') f(\mathbf{x}') \, d\mathbf{x}'. \tag{83}$$

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This is an *extension* of f from $\mathbb{R}^{n+1}$ to $\mathbb{C}^{n+1}$, since

$$\mathbf{y} \to \mathbf{0} \;\Rightarrow\; \tilde{\delta}(\mathbf{x}+i\mathbf{y}-\mathbf{x}') \to \delta(\mathbf{x}-\mathbf{x}') \;\Rightarrow\; \tilde{f}(\mathbf{x}+i\mathbf{y}) \to f(\mathbf{x}). \tag{84}$$

Now

$$\gamma(-\mathbf{z}) = \gamma(\mathbf{z}) \;\Rightarrow\; \phi(-\mathbf{z}) = \phi(\mathbf{z}) \;\Rightarrow\; \tilde{\delta}(-\mathbf{z}) = \tilde{\delta}(\mathbf{z}),$$

hence

$$\tilde{f}(\mathbf{x}+i\mathbf{y}) = \int_{\mathbb{R}^{n+1}} \tilde{\delta}(\mathbf{x}'-\mathbf{x}-i\mathbf{y})\, f(\mathbf{x}')\, d\mathbf{x}' = \int_{\mathbb{R}^{n+1}} \tilde{\delta}(\mathbf{x}'-i\mathbf{y})\, f^{\mathbf{x}}(\mathbf{x}')\, d\mathbf{x}' = \langle\, \tilde{\delta}, f^{\mathbf{x}}\, \rangle, \tag{85}$$

where $f^{\mathbf{x}}$ is the test function defined by

$$f^{\mathbf{x}}(\mathbf{x}') = f(\mathbf{x}+\mathbf{x}') = f(\boldsymbol{x}+\boldsymbol{x}', s+s').$$

Note that in (85), it is $\tilde{\delta}(\mathbf{x}'-i\mathbf{y})$ rather than $\tilde{\delta}(\mathbf{x}'+i\mathbf{y})$ that acts on $f^{\mathbf{x}}$. We will account for this in Equation (86) below by letting $i \to -i$ in (82). Thus, since $\mathbf{z} = \mathbf{x}+i\mathbf{y} = (\boldsymbol{x}, s+it)$, (81) gives

$$\tilde{f}(\boldsymbol{x}, s+it) = \frac{t\sqrt{\pi}}{\Gamma(k+\frac{1}{2})}\, D_r^k F^{\mathbf{x}}(r)\,\Big|_{r=t}, \qquad F^{\mathbf{x}}(r) = r^{2k-1}\left[\bar{f}^{\mathbf{x}}(r,0) + i\,\frac{r^2-t^2}{2kt}\,\bar{f}_{s'}^{\mathbf{x}}(r,0)\right] \tag{86}$$

with

$$\bar{f}^{\mathbf{x}}(r,s') \equiv \int_{S^{n-1}} f^{\mathbf{x}}(r\boldsymbol{\sigma}+s'\hat{\mathbf{y}})\, d\boldsymbol{\sigma} = \int_{S^{n-1}} f(\mathbf{x}+r\boldsymbol{\sigma}+s'\hat{\mathbf{y}})\, d\boldsymbol{\sigma} = \int_{S^{n-1}} f(\boldsymbol{x}+r\boldsymbol{\sigma}, s+s')\, d\boldsymbol{\sigma}.$$

Therefore

$$\bar{f}^{\mathbf{x}}(r,0) = \int_{S^{n-1}} f(\boldsymbol{x}+r\boldsymbol{\sigma}, s)\, d\boldsymbol{\sigma} \tag{87}$$

is the mean of f over the sphere of radius r centered at $\mathbf{x}$ and orthogonal to $\mathbf{y}$, and

$$\bar{f}_{s'}^{\mathbf{x}}(r,0) \equiv \partial_{s'} \bar{f}^{\mathbf{x}}(r,s')\,\Big|_{s'=0} = \int_{S^{n-1}} f_s(\boldsymbol{x}+r\boldsymbol{\sigma}, s)\, d\boldsymbol{\sigma} \equiv \bar{f}_s^{\mathbf{x}}(r,0) \tag{88}$$

is the mean of f_s over the same sphere.

Theorem 5 *Let $n = 2k+1 \geq 3$ and $f \in C^{k+2}(\mathbb{R}^{n+1})$. Then $\tilde{f}(\boldsymbol{x}, s+it)$ belongs to $C^2(\mathbb{R}^{n,1})$ as a function of $(\boldsymbol{x}, t)$, and it is the unique "classical" solution to the following Cauchy problem for the wave equation:*

$$\partial_t^2\, \tilde{f}(\boldsymbol{x}, s+it) = \Delta_{\boldsymbol{x}}\, \tilde{f}(\boldsymbol{x}, s+it) \tag{89a}$$

$$\lim_{t\to 0} \tilde{f}(\boldsymbol{x}, s+it) = f(\boldsymbol{x}, s), \quad \lim_{t\to 0} \partial_t \tilde{f}(\boldsymbol{x}, s+it) = i f_s(\boldsymbol{x}, s). \tag{89b}$$

Furthermore, if $f(\boldsymbol{x}, s)$ can be continued analytically in s, then $\tilde{f}(\boldsymbol{x}, s+it)$ is that continuation.

Proof: The first of the initial conditions is just the extension property (84), which has already been established. In view of this, the second initial condition can be written as

$$\lim_{t\to 0} (\partial_s + i\partial_t)\, \tilde{f}(\boldsymbol{x}, s+it) = 0, \tag{90}$$

which is the *Cauchy-Riemann equation* at $t = 0$. If (90) were to hold in a complex neighborhood U of s_0, it would imply that $\tilde{f}(\boldsymbol{x}, s+it)$ is holomorphic in $s+it \in U$. Since analytic continuation is unique when it exists, this proves our claim that $\tilde{f}(\boldsymbol{x}, s+it)$ is the analytic continuation of $f(\boldsymbol{x}, s)$ whenever the latter exists.

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Without assuming analyticity, we now prove that $\tilde{f}(\boldsymbol{x}, s + it)$ solves the above Cauchy problem. Let

$$u(\boldsymbol{x}, t) = \tilde{f}(\boldsymbol{x}, s + it), \quad v(\boldsymbol{x}, r) = \bar{f}^{\boldsymbol{x}}(r, 0), \quad w(\boldsymbol{x}, r) = i\bar{f}_s^{\boldsymbol{x}}(r, 0), \tag{91}$$

where the dependence on the parameter s is suppressed. Then (86) becomes

$$u(\boldsymbol{x}, t) = \frac{t\sqrt{\pi}}{\Gamma(k + \frac{1}{2})} D_r^k \left[r^{2k-1} v(\boldsymbol{x}, r) + \frac{r^2 - t^2}{2kt} r^{2k-1} w(\boldsymbol{x}, r) \right]_{r=t}. \tag{92}$$

According to (87), $v(\boldsymbol{x}, r)$ is the mean of f over the sphere of radius r centered at $\boldsymbol{x}$, hence $v(\boldsymbol{x}, 0) = f(\boldsymbol{x}, s)$. Similarly, (88) states that $w(\boldsymbol{x}, r)$ is the mean of if_s over the same sphere, so $w(\boldsymbol{x}, 0) = if_s(\boldsymbol{x}, s)$. We therefore need to show that u solves the following Cauchy problem:

$$u_{tt}(\boldsymbol{x}, t) = \Delta u(\boldsymbol{x}, t) \tag{93}$$
$$u(\boldsymbol{x}, 0) = v(\boldsymbol{x}, 0), \quad u_t(\boldsymbol{x}, 0) = w(\boldsymbol{x}, 0).$$

By the definition of D_r, the left equation in (86) is

$$\frac{\sqrt{\pi}}{2^k \Gamma(k + \frac{1}{2})} \partial_t \left(\frac{\partial}{t \partial t} \right)^{k-1} t^{2k-1} v(\boldsymbol{x}, t) = c_n \partial_t \left(\frac{\partial}{t \partial t} \right)^{k-1} t^{2k-1} v(\boldsymbol{x}, t), \tag{94}$$

where

$$c_n = \frac{1}{1 \cdot 3 \cdots (2k - 1) = 1 \cdot 3 \cdots (n - 2)}.$$

The right equation in (86) is

$$c_n \left(\frac{\partial}{r \partial r} \right)^k \left[\frac{r^2 - t^2}{2k} r^{2k-1} w(\boldsymbol{x}, r) \right]_{r=t}. \tag{95}$$

Letting $\xi = r^2/2$, a straightforward computation shows that for any function $G(\xi)$,

$$\left(\frac{\partial}{r \partial r} \right)^k \left[(r^2 - t^2) G(\xi) \right] = \partial_\xi^k \left[(2\xi - t^2) G(\xi) \right] = 2k\partial_\xi^{k-1} G(\xi) + (2\xi - t^2) \partial_\xi^k G(\xi)$$

$$= 2k \left(\frac{\partial}{r \partial r} \right)^{k-1} G + (r^2 - t^2) \left(\frac{\partial}{r \partial r} \right)^k G.$$

Thus (95) becomes

$$c_n \left(\frac{\partial}{r \partial r} \right)^{k-1} \left[r^{2k-1} w(\boldsymbol{x}, r) \right]_{r=t} = c_n \left(\frac{\partial}{t \partial t} \right)^{k-1} t^{2k-1} w(\boldsymbol{x}, t). \tag{96}$$

The sum of (94) and (96) is precisely the solution $u(\boldsymbol{x}, t)$ of the initial-value problem (93) with $n = 2k+1$, as expressed in terms of *spherical means*. See John [J55], Courant and Hilbert [CH62, pp. 699–703] or Folland [F95, p. 170].

That $u \in C^2(\mathbb{R}^{n,1})$ follows because $v(\boldsymbol{x}, r) \in C^{k+2}(\mathbb{R}^n)$ in $\boldsymbol{x}$ and (94) contains k derivatives, while $w(\boldsymbol{x}, r) \in C^{k+1}(\mathbb{R}^n)$ in $\boldsymbol{x}$ and (96) contains $k-1$ derivatives. Finally, uniqueness of the solution u is a general property of the Cauchy problem. ∎

As mentioned earlier, the above result can be extended to an *even* number n of space dimensions by applying the recursion relation (70) between the singular source distributions in $\mathbb{C}^n$ and $\mathbb{C}^{n+1}$. In terms of the solutions $\tilde{f}(\boldsymbol{x}, s + it)$ to the Cauchy problem, this amounts to using Hadamard's *method of descent* [H52, CH62]. Consequently, *the same formula (85) gives the solution of the Cauchy problem for the wave equation in $\mathbb{R}^{n,1}$ for all values of $n \geq 2$.*

The support properties of $\tilde{\delta}$ now imply some important attributes of waves in $\mathbb{R}^{n,1}$. For $\mathbf{z} = \mathbf{x} + i\mathbf{y} \in \mathbb{C}^{n+1}$ with $\mathbf{y} \neq \mathbf{0}$, recall that

$$\operatorname{supp} \tilde{\delta}(\mathbf{x} + i\mathbf{y}) = \begin{cases} \mathcal{B}(\mathbf{y}) & \text{for odd } n \geq 3 \\ E_0(\mathbf{y}) & \text{for all other } n \geq 2. \end{cases}$$

From (85) we can therefore immediately draw the following conclusions about *waves* (solutions of the wave equation) $u(\boldsymbol{x}, t)$ in n space dimensions:

- For odd $n \geq 3$, $u(\boldsymbol{x}, t)$ depends on the values of $u(\boldsymbol{x} + \boldsymbol{v}, 0)$ and $u_t(\boldsymbol{x} + \boldsymbol{v}, 0)$ only in an arbitrarily thin shell containing the sphere $|\boldsymbol{v}| = t$. (We need a shell, rather than the sphere itself, because of the derivatives appearing in (94) and (96).) This is the strong form of *Huygens' principle* [BC87], which states that u depends on the initial data only on the *light cone.*

- For all other $n \geq 2$, $u(\boldsymbol{x}, t)$ depends on the values of $u(\boldsymbol{x} + \boldsymbol{v}, 0)$ and $u_t(\boldsymbol{x} + \boldsymbol{v}, 0)$ in the *past cone* $|\boldsymbol{v}| \leq t$. This is the *principle of causality,* which states that no signal (information, energy) can travel with speed greater than $c = 1$. (If we rescale time by $t \to ct$ with arbitrary $c > 0$, then the maximum propagation speed is c.)

We emphasize that *although holomorphy was needed in the definition of $\tilde{\delta}$, it is not necessary for the above relation to the wave equation.* This distinguishes our results from all similar results in the literature of which I am aware, where holomorphy of the Cauchy data is essential. See Garabedian [G64, pp. 191–202] and Ryan [R90, R90a, R96a].

Theorem 5 and its counterpart for even n can be extended in various ways.

- Clearly it is not necessary to assume that $t > 0$, since the support of $\tilde{\delta}(\mathbf{x} + i\mathbf{y})$ is symmetric with respect to $\mathbf{y} \to -\mathbf{y}$. When $t < 0$, $\tilde{f}(\boldsymbol{x}, s + it)$ is a solution of the "final-value problem" in terms of the Cauchy data at $t = 0$.

- The Cauchy data $f(\mathbf{x})$ need not belong to $C^{k+2}(\mathbb{R}^{n+1})$. When f is a distribution belonging, say, to some Sobolev space [F95, Chapter 6], then $\tilde{f}(\boldsymbol{x}, s + it)$ is a *distributional* solution and the derivatives in (94) and (96) must be interpreted as distributional derivatives.

- To solve the *inhomogeneous* wave equation

$$u_{tt}(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) = j(\boldsymbol{x}, t)$$
$$u(\boldsymbol{x}, 0) = v(\boldsymbol{x}), \quad u_t(\boldsymbol{x}, 0) = w(\boldsymbol{x}),$$

one can apply Duhamel's principle to solutions of the homogeneous equation [F95]. This involves integration of $\tilde{\delta}(\mathbf{z}')$ on the truncated solid light cone with $0 \leq t' \leq t$. However, we will see that in the Clifford setting, the current formalism leads directly to solutions of inhomogeneous hyperbolic equations, where the time-dependent source is determined by the given function $f(\mathbf{x})$ in Euclidean spacetime.

7 Extension to Clifford Analysis

Clifford and quaternionic analyses [BDS82, R96, GS97, R98] are generalizations to $\mathbb{R}^n$ of one-dimensional complex analysis that are proving to be a unifying and very powerful tool in physics [H66, KS96, MM98, O98, B99]. We now show that all the above constructions generalize naturally to this setting. Let $\mathcal{C}\ell_n$ be the *complex* Clifford algebra generated by elements $e_1, \cdots, e_n$ satisfying the anticommutation relations

$$e_k e_l + e_l e_k = 2\delta_{kl} \qquad 1 \leq k, l \leq n. \tag{97}$$

As a complex vector space, $\mathcal{C\ell}_n$ has dimension 2^n with basis vectors

$$e_K \equiv e_{k_1}\cdots e_{k_p}, \quad K = \{k_1, \cdots, k_p\}, \quad 0 \le p \le n, \quad 1 \le k_1 < \cdots < k_p \le n,$$

where the element labeled by the empty set $K = \varnothing$ $(p = 0)$ is by definition $e_\varnothing = 1$. Thus a general vector has the form $v = \sum_K c_K e_K$, where $c_K \in \mathbb{C}$ and the sum runs over the 2^n sets K as above. The element $\sum_{k=1}^n z_k e_k \in \mathcal{C\ell}_n$ is identified with the vector $\boldsymbol{z} \in \mathbb{C}^n$, and by (97) the products and squares of such vectors satisfy

$$\boldsymbol{zw} + \boldsymbol{wz} = 2\sum_{k=1}^n z_k w_k \equiv 2\,\boldsymbol{z}\cdot\boldsymbol{w}, \qquad \boldsymbol{z}^2 = \boldsymbol{z}\cdot\boldsymbol{z} \equiv \gamma(\boldsymbol{z})^2, \tag{98}$$

where $\gamma(\boldsymbol{z})$ is the complex Euclidean distance function. This connection will be our basis for generalizing holomorphic potential theory to the Clifford setting.

Clifford analysis is a noncommutative calculus dealing with Clifford-valued functions

$$f : \mathbb{R}^n \to \mathcal{C\ell}_n, \quad f(\boldsymbol{x}) = \sum_K e_K f_K(\boldsymbol{x}) \quad \text{where} \quad f_K : \mathbb{R}^n \to \mathbb{C}.$$

The primary tool is the *Dirac operator* $\boldsymbol{D} = \sum_k e_k \partial_k$, which is closely related to the exterior derivative [AM78] but in addition incorporates the underlying metric. It acts on Clifford-valued functions from either left or right by

$$\boldsymbol{D}f \equiv \sum_{k=1}^n e_k \frac{\partial f}{\partial x_k} \ne f\overleftarrow{\boldsymbol{D}} \equiv \sum_{k=1}^n \frac{\partial f}{\partial x_k} e_k \tag{99}$$

$$\Rightarrow \boldsymbol{D}^2 f = f\overleftarrow{\boldsymbol{D}}^2 = \sum_K e_K \Delta f_K = \Delta f. \tag{100}$$

Thus $\boldsymbol{D}$ is a "square root" of the Laplacian Δ in $\mathbb{R}^n$. It is an *elliptic* operator because the relations (97) are based on the Euclidean metric in $\mathbb{R}^n$. In 1928, Dirac formulated a similar operator in the Minkowskian spacetime $\mathbb{R}^{3,1}$, where it is *hyperbolic* and its square is the wave operator. This formed the basis for his relativistic wave equation of the electron [D58], which had a revolutionary impact on physics, including especially the dramatic prediction of antimatter. Mathematicians usually prefer Euclidean Dirac operators because, among other things, they yield powerful methods for solving boundary-value problems generalizing those in one-dimensional complex analysis [GS97]. For this and similar reasons, most mathematical work on Dirac operators is restricted to the elliptic case; see the discussion in [O98, p. 1]. Consequently, rigorous analyses of Maxwell's equations by Clifford methods usually deal with static or time-harmonic fields [KS96, MM98]. By assuming that the boundary/Cauchy data is holomorphic, it is possible to arrive at solutions of hyperbolic Dirac equations through analytic continuation [R90, R90a, R96a], generalizing the method employed by Garabedian for establishing a connection between solutions of the Laplace and wave equations [G64]. As in Section 6, the present method is *not* restricted by this assumption. We will "Cliffordize" the holomorphic potential theories in $\mathbb{C}^n$ and $\mathbb{C}^{n+1}$, interpreted respectively as *complex space* and *complex spacetime*. The relation established in Section 6 between Laplacians and wave operators then yields the desired connection between elliptic and hyperbolic Dirac operators.

The Clifford counterpart of the Newtonian potential $\phi(\boldsymbol{x})$ is the *Cauchy kernel* $C : \mathbb{R}^n \to \mathcal{C\ell}_n$, defined as the fundamental solution of $\boldsymbol{D}$:

$$\boldsymbol{D}C(\boldsymbol{x}) = C(\boldsymbol{x})\overleftarrow{\boldsymbol{D}} = \delta(\boldsymbol{x}), \qquad \lim_{r\to\infty} C(\boldsymbol{x}) = 0.$$

Because $\boldsymbol{D}^2 = \Delta$, the solution is easily expressed in terms of the Newtonian potential:

$$\phi = \frac{r^{2-n}}{\omega_n(2-n)}, \qquad C(\boldsymbol{x}) = \boldsymbol{D}\phi(\boldsymbol{x}) = \frac{\boldsymbol{x}}{\omega_n\, r^n}. \tag{101}$$

The expression on the right remains valid for $n = 2$ if we take $\phi = (2\pi)^{-1} \ln r$. Replacing $r(\boldsymbol{x})$ with $\gamma(\boldsymbol{z})$ immediately gives an extension to $\mathbb{C}^n$, where the point source δ becomes the extended source $\tilde{\delta}$:

$$C(\boldsymbol{z}) \equiv \boldsymbol{D}\phi(\boldsymbol{z}) = \phi(\boldsymbol{z})\overleftarrow{\boldsymbol{D}} = \frac{\boldsymbol{z}}{\omega_n \gamma^n}, \quad \boldsymbol{z} \in \mathbb{C}^n, \ n \geq 2 \tag{102}$$

$$\Rightarrow \quad \boldsymbol{D}C(\boldsymbol{z}) = C(\boldsymbol{z})\overleftarrow{\boldsymbol{D}} = \Delta\phi(\boldsymbol{z}) = \tilde{\delta}(\boldsymbol{z}). \tag{103}$$

For all even $n \geq 2$, $C(\boldsymbol{z})$ is holomorphic on the complement of the null cone $\mathcal{N}$. For odd $n \geq 3$, it inherits the branch cut from γ. Thus for all $n \geq 2$, $C(\boldsymbol{z})$ is holomorphic on the complement of the set

$$\mathcal{S}_n = \begin{cases} \{\boldsymbol{x} + i\boldsymbol{y} \mid (\boldsymbol{x} + i\boldsymbol{y})^2 = 0\} & \text{for even } n \geq 2 \\ \{\boldsymbol{x} + i\boldsymbol{y} \in \mathbb{C}^n \mid \boldsymbol{x} \in E_\circ(\boldsymbol{y})\} & \text{for odd } n \geq 3. \end{cases}$$

Given a Clifford-valued test function $f : \mathbb{R}^n \to C\ell_n$, we define its extension to $\tilde{f} : \mathbb{C}^n \to C\ell_n$ exactly as before:

$$\tilde{f}(\boldsymbol{z}) \equiv \int_{\mathbb{R}^n} \tilde{\delta}(\boldsymbol{x}' - \boldsymbol{z}) f(\boldsymbol{x}') \, d\boldsymbol{x}', \quad \Rightarrow \quad \lim_{\boldsymbol{y} \to \boldsymbol{0}} \tilde{f}(\boldsymbol{x} + i\boldsymbol{y}) = f(\boldsymbol{x}). \tag{104}$$

Recall that for $\tilde{f}$ to be defined, f must be C^k with $k = \left[\frac{n-1}{2}\right]$ if $n \geq 3$. It can also be shown [K00] that f must be C^1 if $n = 2$. Inserting one of the expressions (103) for $\tilde{\delta}$ and integrating by parts gives

$$\tilde{f}(\boldsymbol{z}) = \int_{\mathbb{R}^n} C(\boldsymbol{x}' - \boldsymbol{z})\overleftarrow{\boldsymbol{D}} f(\boldsymbol{x}') \, d\boldsymbol{x}' = -\int_{\mathbb{R}^n} C(\boldsymbol{x}' - \boldsymbol{z})\boldsymbol{D} f(\boldsymbol{x}') \, d\boldsymbol{x}', \tag{105}$$

where $\overleftarrow{\boldsymbol{D}}$ denotes the left-acting Dirac opeator with respect to $\boldsymbol{x}'$. We will use this expression to derive an extended version of the *Borel-Pompeiu formula*. Let M be a bounded domain in $\mathbb{R}^n$ with piecewise smooth (C^1) boundary ∂M, and let

$$f_M(\boldsymbol{x}) = \chi_M(\boldsymbol{x}) f(\boldsymbol{x}), \quad \text{where} \quad \chi_M(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in M \\ 0 & \text{if } \boldsymbol{x} \in \bar{M}' \equiv \mathbb{R}^n \backslash \bar{M}. \end{cases}$$

We do not need to define f_M on ∂M. Taking the *distributional* derivatives of f_M gives

$$\boldsymbol{D}f_M(\boldsymbol{x}) = (\boldsymbol{D}\chi_M(\boldsymbol{x}))f(\boldsymbol{x}) + \chi_M(\boldsymbol{x})\boldsymbol{D}f(\boldsymbol{x}). \tag{106}$$

We want to substitute this into (105) to obtain an expression for the extension $\tilde{f}_M(\boldsymbol{z})$ of $f_M(\boldsymbol{x})$. This will make sense if the singularities of $\boldsymbol{D}f_M$ do not meet the singularities of $C(\boldsymbol{x} - \boldsymbol{z})$. That will be the case if we assume that

$$\boldsymbol{z} \notin \partial M + \mathcal{S}_n \equiv \{\boldsymbol{x}_b + \boldsymbol{z} \mid \boldsymbol{x}_b \in \partial M, \ \boldsymbol{z} \in \mathcal{S}_n\}, \tag{107}$$

and we refer to such points $\boldsymbol{z} \in \mathbb{C}^n$ as *regular* with respect to ∂M. Note that when $\boldsymbol{z} \in \mathbb{R}^n$, this means simply that $\boldsymbol{z} \notin \partial M$. For $\boldsymbol{z}$ regular, we may substitute (106) into (105) to obtain

$$\tilde{f}_M(\boldsymbol{z}) = -\int_{\mathbb{R}^n} C(\boldsymbol{x}' - \boldsymbol{z})(\boldsymbol{D}\chi_M(\boldsymbol{x}'))f(\boldsymbol{x}') \, d\boldsymbol{x}' - \int_{\mathbb{R}^n} C(\boldsymbol{x}' - \boldsymbol{z})\chi_M(\boldsymbol{x}')\boldsymbol{D}f(\boldsymbol{x}') \, d\boldsymbol{x}'. \tag{108}$$

We claim that the first term can be written in classical (non-distributional) form as

$$\int_{\partial M} C(\boldsymbol{x}' - \boldsymbol{z}) \, \boldsymbol{n}(\boldsymbol{x}')f(\boldsymbol{x}') \, d\sigma(\boldsymbol{x}'), \tag{109}$$

where $\boldsymbol{n}(\boldsymbol{x})$ is the outgoing unit normal at $\boldsymbol{x} \in \partial M$ and $d\sigma(\boldsymbol{x})$ is the area measure on ∂M induced from the volume measure $d\boldsymbol{x}$. To see this, note that there exists a differentiable function $\mu(\boldsymbol{x})$ such that

$$M = \{\boldsymbol{x} \mid \mu(\boldsymbol{x}) > 0\}, \quad \partial M = \{\boldsymbol{x} \mid \mu(\boldsymbol{x}) = 0\}, \quad \nabla\mu(\boldsymbol{x}) = -|\nabla\mu(\boldsymbol{x})| \, \boldsymbol{n}(\boldsymbol{x}).$$

Then χ_M can be expressed in terms of the Heaviside step function θ by $\chi_M(\boldsymbol{x}) = \theta(\mu(\boldsymbol{x}))$, hence

$$\boldsymbol{D}\chi_M(\boldsymbol{x}) = \delta(\mu(\boldsymbol{x}))\boldsymbol{D}\mu(\boldsymbol{x}) = \delta(\mu(\boldsymbol{x}))\nabla\mu(\boldsymbol{x}) = -\delta(\mu(\boldsymbol{x}))\,|\nabla\mu(\boldsymbol{x})|\,\boldsymbol{n}(\boldsymbol{x}).$$

The connection to the expression (109) can now be made by using the implicit function theorem with $\mu(\boldsymbol{x})$ as one of the local coordinates. The second term in (108) reduces to the integral over M, giving the following result.

Theorem 6 (Extended Borel-Pompeiu formula)

Let $\boldsymbol{z} \in \mathbb{C}^n$ be regular with respect to ∂M. Then

$$\tilde{f}_M(\boldsymbol{z}) = \int_{\partial M} C(\boldsymbol{x}' - \boldsymbol{z})\,\boldsymbol{n}(\boldsymbol{x}')f(\boldsymbol{x}')\,d\sigma(\boldsymbol{x}') - \int_M C(\boldsymbol{x}' - \boldsymbol{z})\,\boldsymbol{D}f(\boldsymbol{x}')\,d\boldsymbol{x}'. \quad \blacksquare \tag{110}$$

For $\boldsymbol{z} \to \boldsymbol{x} \in \mathbb{R}^n \backslash \partial M$, Equation (104) applied to f_M reproduces $f_M(\boldsymbol{x})$ since $\tilde{\delta}(\boldsymbol{x}' - \boldsymbol{x}) = \delta(\boldsymbol{x}' - \boldsymbol{x})$. Thus (110) reduces to the usual Borel-Pompeiu formula for Clifford-valued functions [GS97],

$$\int_{\partial M} C(\boldsymbol{x}' - \boldsymbol{x})\,\boldsymbol{n}(\boldsymbol{x}')f(\boldsymbol{x}')\,d\sigma(\boldsymbol{x}') - \int_M C(\boldsymbol{x}' - \boldsymbol{x})\,\boldsymbol{D}f(\boldsymbol{x}')\,d\boldsymbol{x}' = \begin{cases} f(\boldsymbol{x}), & \boldsymbol{x} \in M \\ 0, & \boldsymbol{x} \in \bar{M}'. \end{cases} \tag{111}$$

We may interpret $f(\boldsymbol{x})$ as a field generated by the source function

$$j(\boldsymbol{x}) \equiv \boldsymbol{D}f(\boldsymbol{x}). \tag{112}$$

Then (111) solves the boundary-value problem for (112), expressing the field f inside M in terms of its sources in M and its values on ∂M. (To investigate the limit of (111) as $\boldsymbol{x} \to \partial M$, one also needs the *Plemelj-Sokhotzki formula* [GS97].) In particular, f is said to be *left-monogenic* in M if $j(\boldsymbol{x}) = 0$ in M. In that case, the second term in (111) vanishes and the Borel-Pompeiu formula reduces to a multidimensional generalization of Cauchy's integral formula, with monogenicity replacing holomorphy.

To describe waves, such as a time-dependent electromagnetic field, we ascend to $\mathbb{C}^{n+1}$ as explained earlier. In the notation of Section 6, let

$$\boldsymbol{z} = \boldsymbol{x} + i\boldsymbol{y} \in \mathbb{C}^n, \quad \mathbf{z} = (\boldsymbol{z}, z_0) = \mathbf{x} + i\mathbf{y} \in \mathbb{C}^{n+1}$$
$$\mathbf{y} = (\mathbf{0}, t) \quad \mathbf{x} = (\boldsymbol{x}, s) \;\Rightarrow\; \mathbf{z} = (\boldsymbol{x}, s + it).$$

The generators of $C\ell_{n+1}$ are $\{e_0, \cdots, e_n\}$, satisfying relations identical to (97) but with $0 \le k, l \le n$. The elliptic Dirac operators in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ are

$$\boldsymbol{D} = \sum_{k=1}^n e_k \partial_k, \quad \mathbf{D} \equiv \sum_{k=0}^n e_k \partial_k = \boldsymbol{D} + e_0 \partial_s, \qquad \partial_0 \equiv \partial_s.$$

We also define the *hyperbolic* (space-time) Dirac operator by formally substituting $s \to it$ in $\mathbf{D}$:

$$\tilde{\mathbf{D}} \equiv \boldsymbol{D} - ie_0 \partial_t.$$

Then

$$\mathbf{D}^2 = \Delta + \partial_s^2 \quad \text{and} \quad \tilde{\mathbf{D}}^2 = \Delta - \partial_t^2 = \Box,$$

where Δ is the spatial Laplacian in $\mathbb{R}^n$ and $\Box$ is the wave operator in $\mathbb{R}^{n,1}$.

Theorem 7 (Inhomogeneous Dirac/Maxwell Equation) *For $n = 2k + 1 \ge 3$, let $f : \mathbb{R}^{n+1} \to C\ell_{n+1}$ be a C^{k+2} function and define the functions $j : \mathbb{R}^{n+1} \to C\ell_{n+1}$ and $\tilde{j} : \mathbb{C}^{n+1} \to C\ell_{n+1}$ by*

$$\mathbf{D}f(\boldsymbol{x}, s) = j(\boldsymbol{x}, s), \qquad \tilde{\mathbf{D}}\tilde{f}(\boldsymbol{x}, s + it) = \tilde{j}(\boldsymbol{x}, s + it). \tag{113}$$

Then $\tilde{j}(\boldsymbol{x}, s + it)$ is a C^1 solution of

$$\tilde{\mathbf{D}}\tilde{j}(\boldsymbol{x}, s + it) = 0, \tag{114}$$

and the system $\{\tilde{f}, \tilde{j}\}$ satisfies the initial conditions

$$\tilde{f}(\boldsymbol{x}, s) = f(\boldsymbol{x}, s), \qquad \tilde{j}(\boldsymbol{x}, s) = j(\boldsymbol{x}, s). \tag{115}$$

Proof: This follows directly from Theorem 5. Since $\tilde{f}(\boldsymbol{x}, s + it)$ is a C^2 solution of the homogeneous wave equation, $\tilde{j}(\boldsymbol{x}, s + it)$ is C^1 and

$$\tilde{\mathbf{D}}\tilde{j} = \tilde{\mathbf{D}}^2\tilde{f} = \Box\tilde{f} = 0.$$

From the initial conditions on $\tilde{f}$, we have $\tilde{f}(\boldsymbol{x}, s) = f(\boldsymbol{x}, s)$ and

$$\lim_{t \to 0} \tilde{j}(\boldsymbol{x}, s + it) = \boldsymbol{D}\tilde{f}(\boldsymbol{x}, s) - ie_0 \lim_{t \to 0} \partial_t \tilde{f}(\boldsymbol{x}, s + it)$$
$$= \boldsymbol{D}f(\boldsymbol{x}, s) + e_0\partial_s f(\boldsymbol{x}, s) = \boldsymbol{D}f(\boldsymbol{x}, s) = j(\boldsymbol{x}, s). \quad \blacksquare$$

For $n = 3$, the right side of (113) is precisely the Clifford form of the *inhomogeneous, time-dependent Maxwell equations* [H66, O98, B99], where

$$\tilde{f}(\boldsymbol{x}, s + it) = \sum_{0 = \mu < \nu}^{3} e_{\mu\nu} F_{\mu\nu}(\boldsymbol{x}, t) \equiv F(\boldsymbol{x}, t) \tag{116}$$

is the electromagnetic field (a bivector),

$$\tilde{j}(\boldsymbol{x}, s + it) = \sum_{\mu=0}^{3} e_\mu J_\mu(\boldsymbol{x}, t) \equiv J(\boldsymbol{x}, t) \tag{117}$$

is the charge-current density (a four-vector), and the Euclidean time s is a free parameter. As with our distance function, the Minkowskian metric appears in (116) and (117) when the time components J_0 and F_{0k} $(k = 1, 2, 3)$ are imaginary and the spatial components are real.

Again, we did not need to assume that $\tilde{f}$ or $\tilde{j}$ are holomorphic in $s + it$. Note that although $\tilde{j}(\boldsymbol{x}, s + it)$ is a time-evolved extension of $j(\boldsymbol{x}, s)$, it can be shown [K00] that it is *not* the one obtained by convolving with $\tilde{\delta}$. Instead, the time dependence of the charge-current density is governed by (114), the scalar part of which is the *continuity equation* implying conservation of the total charge.

In (113) and (114), we have taken $M = \mathbb{R}^{n+1}$ for simplicity, giving a pure initial-value problem. These equations may also be formulated in a bounded region $M \subset \mathbb{R}^{n+1}$ by using the extended Borel-Pompeiu formula for $\tilde{f}_M(\boldsymbol{x}, s + it)$. This suggests that Theorem 6 may be used to solve *mixed initial/boundary value problems*, provided one is careful about dealing with the singular points $(\boldsymbol{x}, s + it) \in \partial M + \mathcal{S}_{n+1}$. These are in the domain of influence of the boundary; that is, they can be reached at time t by signals originating from the boundary ∂M at time $t = 0$. However, this still does not give a mechanism for boundary effects initiated at $t > 0$ (such as reflections) to influence the solution. This and related questions will be treated elsewhere.

Acknowledgements

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